USA Mathematical Talent Search
Expected Value
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1 Introduction

You are probably already familiar with the concept of *mean* or *average*. For example, given the three numbers 5, 8, and 14, their average is $(5+8+14)/3 = 27/3 = 9$. More generally, if we are given numbers $x_1, x_2, \ldots, x_n$, then their average, often denoted $\bar{x}$, is:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

If we extend the concept of average to the outcome of a random event, we get what is called *expected value* or *expectation*. Expected value is the term we use to indicate the average result we would expect to get if we did a large number of trials for any experiment. Essentially, expected value is a weighted average, in which the more likely the outcome, the more that outcome is weighted.

Let's look at a simple example. Suppose that we are flipping a coin. If we get heads, we win \$2, but if we get tails, we lose \$1. If we flip the coin 1,000 times, we would expect to get heads 500 times and tails 500 times. We would win a total of \$1,000 from our 500 heads and lose a total of \$500 from our 500 tails, for a total net profit of \$1,000 − \$500 = \$500. Since this profit is realized over the course of 1,000 flips, our average profit per flip is

$$\frac{\$500}{1000} = \$0.50.$$

We say that the expected value of each flip is \$0.50. In this case, since heads and tails are equally likely, the expected value is just the usual average of the two outcomes:

$$\frac{+\$2 + (-\$1)}{2} = \frac{\$1}{2} = \$0.50.$$

However, if we are calculating the expected value of an event in which different outcomes occur with different probability, then we have to take a weighted average of the outcomes, as we will see a bit later.

2 Definition

We'll start by stating the formal definition, but don't be alarmed – the idea of expected value is not nearly as complicated as it may first appear.

Suppose that we have an event in which every outcome corresponds to some value. (An example is rolling a die: the "value" is the number showing on the top face of the die.) We have a list of possible values: $x_1, x_2, \ldots, x_n$. Value x_1 occurs with probability p_1, value x_2 occurs with probability p_2, and so on. Note that

$$p_1 + p_2 + \cdots + p_n = 1,$$

since the probabilities must total to 1. Then the *expected value* of the outcome is defined as the sum of the products of the outcomes' values and their respective probabilities:

$$E = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n.$$

Traditionally, expected value is denoted by the capital letter E or by the Greek letter μ (called "mu").

Let's go back to our coin-flipping example, in which we win \$2 for heads but lose \$1 for tails. Since the probability of heads is $\frac{1}{2}$ and the probability of tails is also $\frac{1}{2}$, we can use the above formula to get:

$$E = \frac{1}{2}(\$2) + \frac{1}{2}(-\$1) = \$1 - \$0.50 = \$0.50,$$

which is what we got before.

3 Some simple examples

Now we'll do some fairly straightforward expected value computations. In all of these computations, we are simply taking the weighted average of the possible outcomes, in which the value of each outcome is weighted by the probability of that outcome.

Problem 1. What is the expected value of the roll of a standard 6-sided die?

Each outcome of rolling a 6-sided die has probability $\frac{1}{6}$, and the possible outcomes are ⚀, ⚁, ⚂, ⚃, ⚄, and ⚅. So the expected value is

$$\frac{1}{6}(1) + \frac{1}{6}(2) + \frac{1}{6}(3) + \frac{1}{6}(4) + \frac{1}{6}(5) + \frac{1}{6}(6) = \frac{21}{6} = 3.5.$$

Note that since each outcome in Problem 1 is equally likely, we can get the expected value simply by averaging the possible outcomes:

$$\frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5.$$

Also note that we can't actually roll a 3.5 on any individual roll. Thus, the expected value is not necessarily the most likely value, but rather the probability-weighted average of all possible values.

Problem 2. Suppose you have a weighted coin in which heads comes up with probability $\frac{3}{4}$ and tails with probability $\frac{1}{4}$. If you flip heads, you win \$2, but if you flip tails, you lose \$1. What is the expected value of a coin flip?

By definition, we multiply the outcomes by their respective probabilities, and add them up:

$$E = \frac{3}{4}(+\$2) + \frac{1}{4}(-\$1) = \$1.50 - \$0.25 = \$1.25.$$

Another way to think of the expected value in Problem 2 is to imagine flipping the coin 1,000 times. Based on the probabilities, we would expect to flip heads 750 times and to flip tails 250 times. We would then win \$1,500 from our heads but lose \$250 from our tails, for a net profit of \$1,250. Since this occurs over the course of 1,000 flips, our average profit per flip is

$$\frac{\$1{,}250}{1{,}000} = \$1.25.$$

Problem 3. In an urn, I have 20 marbles: 2 red, 3 yellow, 4 blue, 5 green, and 6 black. I select one marble at random from the urn, and I win money based on the following chart:

Color	Red	Yellow	Blue	Green	Black
Amount won	\$10	\$5	\$2	\$1	\$0

What is the expected value of my winnings?

I draw a red marble with probability $\frac{2}{20}$, a yellow marble with probability $\frac{3}{20}$, and so on. Therefore, the expected value is

$$\frac{2}{20}(\$10) + \frac{3}{20}(\$5) + \frac{4}{20}(\$2) + \frac{5}{20}(\$1) + \frac{6}{20}(\$0) = \frac{\$48}{20} = \$2.40.$$

One common use of the expected value in a problem like Problem 3 is to determine a *fair price* to play the game. The fair price to play is the expected winnings, which is \$2.40. In the long run, if I were charging people the fair price to play the game, I would expect to break even, neither making nor losing money. If I were running this game at a carnival, I could charge carnival-goers \$2.50 to play the game, and I would expect to make a 10-cent profit, on average, from each person who plays.

Problem 4. Suppose I have a bag with 12 slips of paper in it. Some of the slips have a 2 on them, and the rest have a 7 on them. If the expected value of the number shown on a slip randomly drawn from the bag is 3.25, then how many slips have a 2?

We let x denote the number of slips with a 2 written on them. (This is the usual tactic of letting a variable denote what we're trying to solve for in the problem.) Then there are $12 - x$ slips with a 7 on them.

The probability of drawing a 2 is $\frac{x}{12}$ and the probability of drawing a 7 is $\frac{12-x}{12}$, so the expected value of the number drawn is

$$E = \frac{x}{12}(2) + \frac{12-x}{12}(7) = \frac{84-5x}{12}.$$

But we are given that $E = 3.25$, so we have an equation we can solve for x:

$$3.25 = \frac{84-5x}{12}.$$

Solving this equation gives $x = 9$. Thus 9 of the 12 slips have a 2 written on them.

4 Some more challenging problems

Now that we've finished describing expected value, we'll explore some more interesting problems. We'll solve a more challenging expected value problem, then use expected value as a tool to solve other problems.

Problem 5. Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row $GBBGGGBGBGGGBGBGGBGG$ we have $S = 12$. What is the expected value of S for a randomly chosen arrangement of the 20 people? *(Source: American High School Mathematics Examination, now called the AMC 12)*

First, we note that we can treat all the boys as the same, and all the girls as the same. This reduces our problem to designating 7 spots as B (boys' spots) and 13 as G (girls' spots). We then must find the expected value of the number of times a B is next to a G.

We could start with a tedious straightforward approach – look at each possible arrangement and count the number of BG's and GB's for each:

Arrangement	# of BG's and GB's
$BBBBBBBGGGGGGGGGGGGG$	1
$BBBBBBGBGGGGGGGGGGGG$	3
$BBBBBBGGBGGGGGGGGGGG$	3
$BBBBBBGGGBGGGGGGGGGG$	3
$BBBBBBGGGGBGGGGGGGGG$	3

Um, this is going to take forever. Since there are $\binom{20}{7}$ = (a very big number) ways to choose the B places, our list will be very, very long. We need a better way.

Instead of looking at the problem 'arrangement-by-arrangement', we take an 'element-by-element' approach. We look at the first two spots and calculate in what portion of the arrangements they will be BG or GB. Clearly there are 2 ways to make the first two places a boy next to a girl. We must choose 6 of the remaining 18 places to have boys in them, so there are $\binom{18}{6}$ ways to allocate the rest of the places to boys or girls. Therefore, the number of arrangements in which the first two places have a boy and a girl is $2\binom{18}{6}$.

There's nothing special about the first 2 places! By the same reasoning, there are also $2\binom{18}{6}$ ways the second and third places can have one boy and one girl. And same for the third and fourth, and fourth and fifth, and so on! There are 19 such pairs (not 20), so the total number of times we will have a boy next to a girl is $19 \times 2\binom{18}{6}$. To get our expected value, we divide this by the total number of arrangements, which we found earlier is $\binom{20}{7}$:

$$\frac{19 \times 2\binom{18}{6}}{\binom{20}{7}} = \frac{19 \times 2 \times 18! \times 7! \times 13!}{6! \times 12! \times 20!} = \frac{91}{10}.$$

Problem 6. Suppose you and I play a game. You start with 500 dollars and I start with 1000 dollars. We flip a fair coin repeatedly. Each time it comes up heads, I give you a dollar. Each time it comes up tails, you give me a dollar. We continue playing until one of us has all the money. What is the probability you will win this game?

This problem doesn't look like it has anything to do with expected value! But when we look at the problem through the lens of expected value, we find a slick solution.

The expected value of each coin toss is 0, because it is a fair coin. Since the expected value of each toss is 0, the expected value of any number of tosses is 0. Most importantly, no matter how long the game lasts, the expected value of the whole game is 0 for us both. Since your two possible outcomes are +\$1000 and −\$500, we can now write a simple expression for the expected value of the game for you:

$$E(\text{the game for you}) = (+\$1000)(p) + (-\$500)(1-p).$$

Since your expected value is 0, we have $0 = (+\$1000)(p) + (-\$500)(1-p)$, so $p = 1/3$. Therefore, the probability you will win the game is 1/3.

Expected value can also be cleverly used for existence problems. For example, suppose that the expected value of the American Invitational Mathematics Exam (AIME) score of a USAMTS Gold Prize winner chosen at random is 9.4. From this, we can deduce that at least one of the Gold Prize winners scored a 10 or higher, and we can infer that at least one of them scored a 9 or lower. Therefore, expected value can sometimes be used to solve existence problems of the variety, "Show that there exists one item with at least ..." or "Show that there exists one item with at most ...."

We'll illustrate this strategy with an example.

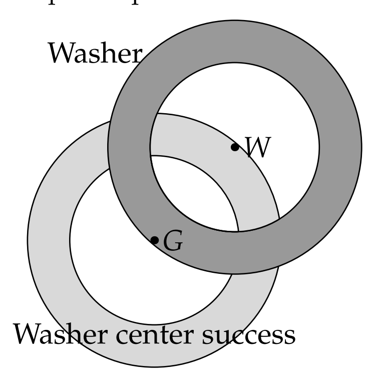
Problem 7. There are 650 special points inside a circle of radius 16. You have a flat washer in the shape of an annulus (the region between two concentric circles), which has an inside radius of 2 and an outside radius of 3. Show that it is always possible to place the washer so that it covers up at least 10 of the special points. *(Source: Problem Solving Strategies by Arthur Engel. Solution by Ravi Boppana, USAMTS supporter.)*

This is an existence problem; we wish to show that there exists some position at which we can place the washer so that it covers at least 10 points. This suggests an expected value approach. If we can show that the expected value of the number of points covered by a washer placed at random is greater than 9, then we can conclude that there must be some place we can put the washer to cover 10 or more points.

Calculating this expected value, however, doesn't appear so simple – for starters, there are infinitely many possible places to put the washer! In probability problems in which the possible range is continuous, we typically reach for the tools of geometry to measure the possible region and the desired region. So, we'll try that here. But how?

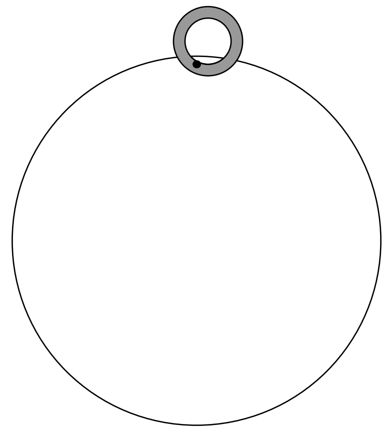
Evaluating each possible position of the washer will literally take forever, since there are an infinite number of places we can put it. However, there are only 650 points in the circle we have to consider. Therefore, we take an element-by-element approach as we did in the BG-arrangements problem. We calculate the portion of the washer placings in which each special point is covered.

Let G be one of our special points. G is covered by the washer if the center of the washer is at least 2 units away from the point, but no more than 3 units away. For example, the center of the washer shown is in our 'success' region, so the washer covers our point G. Any washer centered in the light grey region will cover point G.



Therefore, the area of the region in which the center of the washer must be in order to cover G is $3^2\pi - 2^2\pi = 5\pi$.

We must be careful to include those cases in which the center of the washer is just outside our initial circle, but still covering special points near the circumference of the circle, as shown below.



To take into account the possibility of a washer centered outside the circle covering special points inside the circle (an example of which is shown at right), we note that the 'possible' region in which the center of our washer can be placed and still have the washer cover special points is a circle with radius 19, not 16. Now, we can evaluate the probability that a washer placed at random covers a given special point. Out of the $19^2\pi = 361\pi$ area in which we can place the center of our washer to cover portions of our circle with the washer, there is a 5π area in which it can be placed to cover G. Therefore, the probability G is covered is $(5\pi)/(361\pi) = 5/361$.

There's nothing special about the G we examined! For each of the 650 special points, the probability that a randomly placed washer covers the point is 5/361. Therefore, each special point contributes 5/361 to the expected value of the number of special points covered by a randomly placed washer. So, the total expected value of the number of special points covered by a randomly placed washer is

$$650 \cdot \frac{5}{361} \approx 9.003.$$

Since the expected value of the number of special points covered by a randomly placed washer that overlaps some portion of our circle is greater than 9, there must be some placement of the washer which covers at least 10 special points.$\square$

Article by Dr. David Patrick and Richard Rusczyk, two of the managers of the USA Mathematical Talent Search. Some portions of this article are reprinted with permission from the book Art of Problem Solving: Introduction to Counting & Probability by Dr. David Patrick (publication pending).