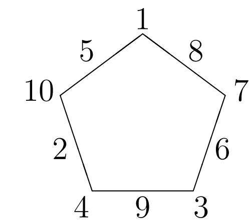
USA Mathematical Talent Search

Solutions to Problem 1/1/16

www.usamts.org

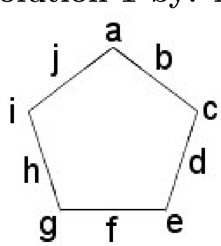
1/1/16. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram on the right shows an arrangement with sum 16. Find, with proof, the smallest possible value for a sum and give an example of an arrangement with that sum.



Credit This problem was invented by George Berzsenyi, inspired by Team Question 1 of the 2001 South East Asian Mathematics Olympiad.

Comments Some students solved this problem by pointing at that 10 must be on at least one side, so $10 + 1 + 2 = 13$ is the smallest possible side sum. Such proofs go on to show that such a side sum is impossible, though this approach has the drawback of requiring many steps that must be discussed. Skipping any step led to a lack of rigor in many solutions. One student found a novel way to examine this approach by grouping the possible sums of 13 into those where the lowest number was 1, 2, or 3.

Solution 1 by: Aaron Pribadi (9/AZ)



The variables a through j are the values arranged around the pentagon.

Since the sums of the sides are the same:

$$a + b + c = c + d + e = e + f + g = g + h + i = i + j + a = \text{the sum of one side}$$

Thus,

$$\frac{(a + b + c) + (c + d + e) + (e + f + g) + (g + h + i) + (i + j + a)}{5} = \text{the sum of one side}$$

or it can be rearranged to

$$\frac{(a + b + c + d + e + f + g + h + i + j) + (a + c + e + g + i)}{5}$$

Since $a + b + c + d + e + f + g + h + i + j$ consists of the sum of the numbers one through 10, it equals $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$, or 55. Thus

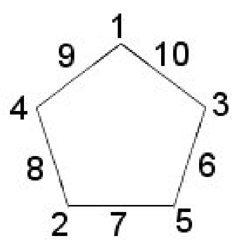
$$\frac{(55) + (a + c + g + e + i)}{5} = 11 + \frac{a + c + g + e + i}{5} = \text{the sum of one side}$$

The numbers are positive, so to minimize the expression above, $a + c + g + e + i$ must be minimized.
It is obvious that the values to minimize the sum $a + c + g + e + i$ are the numbers 1–5 out of the possible numbers of 1 through 10, since those are the 5 smallest numbers. So,

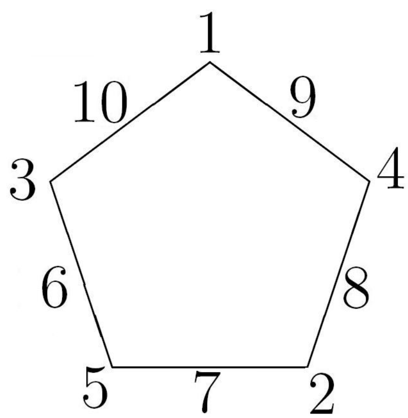
$$11 + \frac{1 + 2 + 3 + 4 + 5}{5} = 11 + 3 = 14 \text{ the minimum sum of one side}$$

14 is the smallest possible value for a sum.

Now that an optimal minimum value has been found, an example will show that the value can be attained. One such example is in the following diagram:



Solution 2 by: Nathan Pflueger (12/WA)
Define S for any arrangement of the numbers 1 through 10 on the edges and vertices of a pentagon to be the sum of the sums of the three numbers along each side of the pentagon. Notice that, in this summation, every number found on an edge will be added once (when the numbers on that edge are added), but each number on a vertex will be added twice (once for each edge connected to that vertex). Thus $S = \Sigma(\text{edge numbers}) + 2\Sigma(\text{vertex numbers})$. However, observe that $\Sigma(\text{edge numbers}) + \Sigma(\text{vertex numbers}) = \Sigma(\text{all numbers})$, and the sum of all numbers is the sum $1 + 2 + \cdots + 10 = 55$, thus $S = 55 + \Sigma(\text{vertex numbers})$. This can be minimized by making the vertex numbers 1 through 5, thus $S \geq 55 + 15 = 70$. If the sum of the numbers on each edge are equal, then each such sum is $S/5$, thus the least possible such sum is $70/5 = 14$. Indeed, this sum is possible by using the arrangement shown below.


1
10
9
3
4
6
8
5
7
2