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Solutions to Problem 1/2/16
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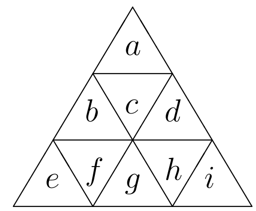
1/2/16. The numbers 1 through 9 can be arranged in the triangles labeled a through i illustrated on the right so that the numbers in each of the 2×2 triangles sum to the same value n; that is

$$a + b + c + d = b + e + f + g = d + g + h + i = n.$$

For each possible sum n, show such an arrangement, labeled with the sum as shown at right. Prove that there are no possible arrangements for any other values of n.

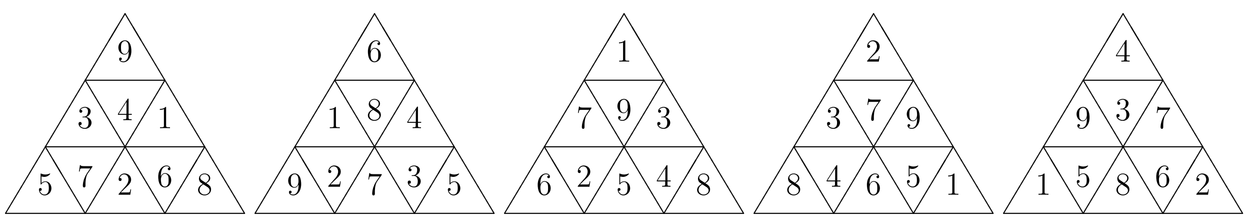


n

Credit This is a take-off on a Hungarian problem that appeared in the book *Brainteasers for Upperclassmen* by Imrecze, Reiman, and Urbán in Hungarian in 1986.

Comments There are basically two steps to the solution: finding bounds on the possible values of n, and then finding which values within those bounds have valid arrangements. Many students simplified the argument by noting the symmetry between arrangements summing to n and arrangements summing to $40 - n$. (Solution 2 below uses this fact.) The major variation amongst different solutions is the method by which the cases $n = 18$ and $n = 22$ were shown to be impossible. Solution 1 shows a nice casework approach. Solution 2 uses a clever observation to eliminate all of the cases at once.

Solution 1 by: Eric Paniagua (12/NY)



17 19 20 21 23

From the problem, we have

$$\begin{aligned} 3n &= a + b + c + d + e + f + g + h + i + (b + d + g) \\ 3n &= 45 + b + d + g \end{aligned}$$

where $\{a, b, c, d, e, f, g, h, i\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Clearly, the maximum and minimum values of this sum are

$$\begin{aligned} \max &= 45 + 7 + 8 + 9 = 69 \\ \min &= 45 + 1 + 2 + 3 = 51 \end{aligned}$$

so we have the bounds on n:

$$\begin{aligned} 51 \leq 3&n \leq 69 \\ 17 \leq \ &n \leq 23 \end{aligned}$$

Example arrangements for $n = 17, 19, 20, 21, 23$ are given above.

Proof that no arrangement exists for $n = 18$.
If $n = 18$ then $b+d+g = 3(18)-45 = 9$ and $\{b, d, g\}$ equals exactly one of $\{4, 2, 3\}, \{1, 5, 3\}, \{1, 2, 6\}$. By the symmetry of the positions of b, d, g in the triangle all assignments of these to the numbers in one of these sets are equivalent.

First, assume $b = 4, d = 2, g = 3$. Then we have

$$n = 18 = a + b + c + d = a + c + 6$$

and $a + c = 12$, so $\{a, c\} = \{7, 5\}$ because this is the only decomposition of 12 not using 2, 3, or 4. Similarly, $e + f = 18 - b - g = 11$ and $\{e, f\} = \{6, 5\}$ contradicting the fact that a, c, e, f are distinct.

Now assume $b = 1, d = 5, g = 3$. We have $a + c = 18 - b - d = 12$, so $\{a, c\} = \{8, 4\}$. Similarly, $h + i = 18 - d - g = 10$ and $\{h, i\}$ equals $\{6, 4\}$ or $\{8, 2\}$ contradicting the fact that a, c, h, i are distinct.

Finally, assume $b = 1, d = 2, g = 6$. Then $a + c = 18 - b - d = 15$, so $\{a, c\} = \{8, 7\}$. Similarly, $e + f = 18 - b - g = 11$ and $\{e, f\}$ equals $\{7, 4\}$ or $\{8, 3\}$ contradicting the fact that a, c, e, f are distinct.

$\therefore n \neq 18$. □

Proof that no arrangement exists for $n = 22$.
If $n = 22$ then $b+d+g = 3(22)-45 = 21$ and $\{b, d, g\}$ equals exactly one of $\{6, 7, 8\}, \{9, 7, 5\}, \{9, 4, 8\}$.

Assume $b = 6, d = 7, g = 8$. Then $a + c = 22 - b - d = 9$, so $\{a, c\} = \{5, 4\}$. Similarly, $e + f = 22 - b - g = 8$ and $\{e, f\} = \{5, 3\}$ contradicting the fact that a, c, e, f are distinct.

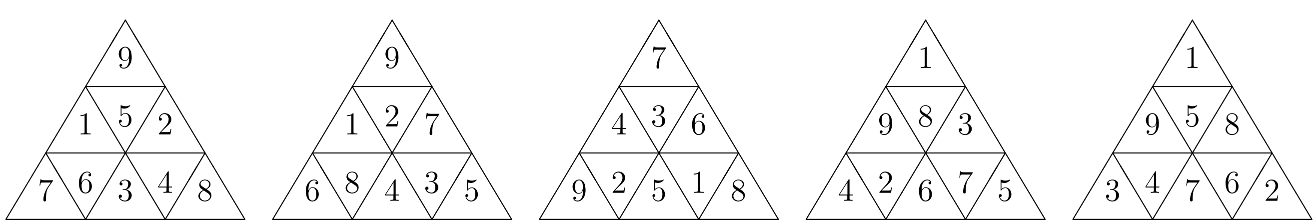
Assume $b = 9, d = 7, g = 5$. Then $a + c = 22 - b - d = 6$, so $\{a, c\} = \{4, 2\}$. Similarly, $e + f = 22 - b - g = 8$ and $\{e, f\} = \{6, 2\}$ contradicting the fact that a, c, e, f are distinct.

Assume $b = 9, d = 4, g = 8$. Then $a + c = 22 - b - d = 9$, so $\{a, c\}$ equals $\{7, 2\}$ or $\{6, 3\}$. Similarly, $e + f = 22 - b - g$ and $\{e, f\} = \{3, 2\}$ contradicting the fact that a, c, e, f are distinct.

$\therefore n \neq 22$. □

Solution 2 by: Adam Hesterberg (10/WA)

Answer: The possible values of n are 17, 19, 20, 21, and 23, as shown below:



17 19 20 21 23

First, we prove that $17 \leq n \leq 23$. Note that

$$
\begin{aligned}
3n &= (a+b+c+d)+(b+e+f+g)+(d+g+h+i) \\
&= \sum(\text{all the numbers})+b+d+g \\
&= 45+b+d+g
\end{aligned}
$$

Since $b+d+g$ is between $1+2+3=6$ and $7+8+9=24$, n is between $\frac{45+6}{3}=17$ and $\frac{45+24}{3}=23$.

Thus, the only values for n left to consider are 18 and 22. Note that if 18 could be constructed, so could 22, by replacing each entry x by $10-x$. Therefore, we need only consider 18.
$n=18$ implies $b+d+g=3*18-45=9$. Without loss of generality, let 9 be a (it will not be b, d, or g since the sum of the other two would then have to be 0). Then,

$$
\begin{aligned}
18 &= a+b+c+d \\
&= 9+c+(b+d+g)-g \\
&= 9+9+c-g \\
0 &= c-g \\
c &= g
\end{aligned}
$$

However, c and g were to be distinct, so this is impossible. Therefore, neither 18 nor 22 can be constructed, so the only possible values for n are the ones constructed above.