USA Mathematical Talent Search

Solutions to Problem 1/3/16

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1/3/16. Given two integers x and y, let $(x\|y)$ denote the *concatenation* of x by y, which is obtained by appending the digits of y onto the end of x. For example, if $x = 218$ and $y = 392$, then $(x\|y) = 218392$.
(a) Find 3-digit integers x and y such that $6(x\|y) = (y\|x)$.
(b) Find 9-digit integers x and y such that $6(x\|y) = (y\|x)$.

Credit The 3-digit variety of the problem was inspired by Problem 28 in the Singapore Mathematical Olympiad (Junior Section) in 2001. The 9-digit extension is due to USAMTS founder Dr. George Berzsenyi.

Comments Many students took a trial-and-error appraoch. The most common algebraic approach to part (a) is reflected in Jason Bland's solution. Many students used this approach for part (b), but a few students used the slick approach of using (a) to get (b) as shown in Nathan Pflueger's solution below. Still others used the number 1,000,001,000,001 as Jason Bland illustrates below. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Nathan Pflueger (12/WA)

(a)
Let $(x, y) = (142, 857)$. Multiplication yields $6(x||y) = 6 \cdot 142857 = 857142 = (y||x)$.

(b)
Let $(x, y) = (142, 857)$ as above. Let $(u, v) = (x||y||x, y||x||y)$. It was shown above that $6(x||y) = (y||x)$ thus $6(u||v) = 6(x||y||x||y||x||y) = (y||x||y||x||y||x) = (v||u)$, thus u and v are the 9-digit integers we seek: 142857142 and 857142857, respectively. Alternating concatenations such as this can also be used to select two such integers for any number of digits of the form $3 + 6n$.

Solution 2 by: Jason Bland (10/PA)

(a) Because x and y each have 3 digits, we can write $(x||y) = 1000x + y$. Therefore, we have

$$\begin{aligned} 6(1000x + y) &= 1000y + x \\ 6000x + 6y &= 1000y + x \\ 5999x &= 994y \\ 857x &= 142y \\ x = 142 & \qquad y = 857 \end{aligned}$$

(b) $(x||y)$ has 6 digits when x and y have 3 digits each and 18 digits when x and y have 9 digits each, so multiplying the equation involving $(x||y)$ and $(y||x)$ for 3-digit x and y by 1,000,001,000,001 gives the equation involving $(x||y)$ and $(y||x)$ for 9-digit x and y.

$$\begin{aligned} 6 * 142{,}857 &= 857{,}142 \\ 6 * 142{,}857{,}142{,}857{,}142{,}857 &= 857{,}142{,}857{,}142{,}857{,}142 \\ x = 142{,}857{,}142 & \qquad y = 857{,}142{,}857 \end{aligned}$$