USA Mathematical Talent Search
Solutions to Problem 1/3/18
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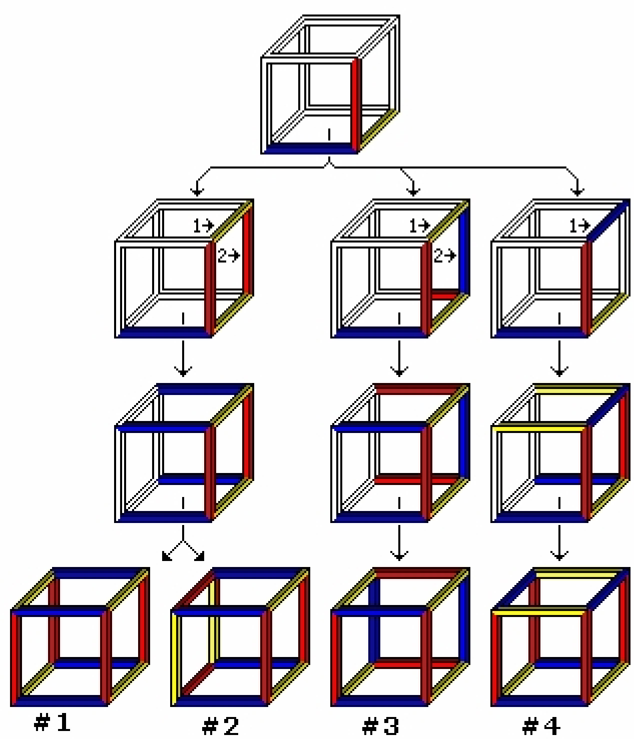
1/3/18. In how many distinguishable ways can the edges of a cube be colored such that each edge is yellow, red, or blue, and such that no two edges of the same color share a vertex? (Two cubes are indistinguishable if they can be rotated into positions such that the two cubes are colored exactly the same.)

Credit This problem was proposed by Richard Rusczyk and Sam Vandervelde.

Comments This problem is best solved by systematic casework and well-drawn diagrams. *Solutions edited by Naoki Sato.*

Solution 1 by: Igor Tolkov (10/WA)

The four possible colorings are as shown.



Without loss of generality, color three mutually adjacent edges blue, red, and yellow as shown in the top layer. Then, consider the edge marked by "1". Because an adjacent edge is red, this edge can be either yellow or blue.

First, assume that it is yellow. Then consider the edge marked by "2". This edge can be either blue or red. If it is red, then all blue edges are uniquely determined and there exist two options for the remaining four edges. This produces two colorings: One with parallel edges of the same color and one with red and yellow edges alternating. If edge 2 is blue, then the other edges are uniquely determined and we obtain a coloring with blue and red edges alternating.

Now, if edge 1 is blue, then the other edges are again uniquely determined and we obtain a coloring with blue and yellow edges alternating. This covers all cases.

Note that all four cubes have a plane of symmetry so orientation does not matter. In other words, each cube can be rotated to account for orientation.