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Solutions to Problem 1/3/19
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1/3/19. We construct a sculpture consisting of infinitely many cubes, as follows. Start with a cube with side length 1. Then, at the center of each face, attach a cube with side length $\frac{1}{3}$ (so that the center of a face of each attached cube is the center of a face of the original cube). Continue this procedure indefinitely: at the center of each exposed face of a cube in the structure, attach (in the same fashion) a smaller cube with side length one-third that of the exposed face. What is the volume of the entire sculpture?

Comments Once the geometry of the sculpture has been determined, the volume can be found by summing an infinite geometric sequence. *Solutions edited by Naoki Sato.*

Solution by: Dmitri Gekhtman (11/IN)

Since the cube with side length 1 has 6 faces, the sculpture has 6 cubes of side length $\frac{1}{3}$ and volume $(\frac{1}{3})^3 = \frac{1}{27}$. Since each of the 6 cubes of side length $\frac{1}{3}$ has 5 exposed faces, there are 6×5 cubes of side length $(\frac{1}{3})^2$ and volume $(\frac{1}{27})^2$. By the same reasoning, there are 6×5^2 cubes of volume $(\frac{1}{27})^n$. In general, the sculpture contains $6 \times 5^{n-1}$ cubes of volume $(\frac{1}{27})^n$, where n is a positive integer. The sculpture contains one cube of volume 1. Therefore, the total volume of the sculpture is

$$1 + \sum_{n=1}^{\infty} 6 \times 5^{n-1} \times \left(\frac{1}{27}\right)^n = 1 + \sum_{n=0}^{\infty} \frac{6}{27} \times \left(\frac{5}{27}\right)^n.$$

Since the infinite sum is an infinite geometric series with first term $\frac{6}{27}$ and common ratio $\frac{5}{27}$, the volume is

$$1 + \frac{\frac{6}{27}}{1 - \frac{5}{27}} = \frac{14}{11}.$$