



USA Mathematical Talent Search

Solutions to Problem 1/4/17

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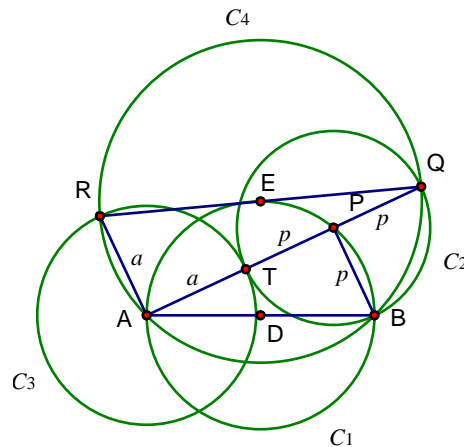
1/4/17. \overline{AB} is a diameter of circle C_1 . Point P is on C_1 such that $AP > BP$. Circle C_2 is centered at P with radius PB . The extension of \overline{AP} past P meets C_2 at Q . Circle C_3 is centered at A and is externally tangent to C_2 . R is on C_3 such that $\overline{AR} \perp \overline{AQ}$. Circle C_4 passes through A , Q , and R . Find, with proof, the ratio between the area of C_4 and the area of C_1 , and show that this ratio is the same for all points P on C_1 such that $AP > BP$.

Credit This problem was proposed by Ismor Fischer, University of Wisconsin.

Comments This problem can be easily solved by labeling lengths, and using right angles to apply Pythagoras's theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: Matt Superdock (9/PA)

Let D be the center of C_1 , and let E be the center of C_4 . Let T be the point of tangency of C_2 and C_3 . Let a be the radius of C_3 , and let p be the radius of C_2 . Then we have $AT = AR = a$ and $PB = PQ = PT = p$.



Since $\overline{AR} \perp \overline{AQ}$, $\angle RAQ$ is a right angle. Since right angles inscribe the diameter of the circle, RQ is the diameter of C_4 . Since $\angle APB$ inscribes \overline{AB} , the diameter of C_1 , $\angle APB$ is also a right angle.

The ratio we are seeking is

$$\begin{aligned} \frac{[C_4]}{[C_1]} &= \frac{\pi(ER)^2}{\pi(AD)^2} = \frac{(QR)^2}{(AB)^2} = \frac{(AR)^2 + (AQ)^2}{(AP)^2 + (BP)^2} = \frac{(AR)^2 + (AT + TP + PQ)^2}{(AT + TP)^2 + (BP)^2} \\ &= \frac{a^2 + (a + p + p)^2}{(a + p)^2 + p^2} = \frac{2a^2 + 4ap + 4p^2}{a^2 + 2ap + 2p^2} = \boxed{2}. \end{aligned}$$

Since we made no assumptions, the ratio is 2 for all points P on C_1 such that $AP > BP$.