1/4/18. Let $S(n) = \sum_{i=1}^{n} (-1)^{i+1} i$. For example, $S(4) = 1 - 2 + 3 - 4 = -2$.
(a) Find, with proof, all positive integers a, b such that $S(a) + S(b) + S(a + b) = 2007$.
(b) Find, with proof, all positive integers c, d such that $S(c) + S(d) + S(c + d) = 2008$.

Credit This problem was proposed by Dave Patrick, and was based on a discussion at the 2006 World Federation of National Mathematics Competitions conference.

Comments Since both parts have the form $S(m)+S(n)+S(m+n)$, it is easiest to analyze this form first to solve for a, b, c and d. *Solutions edited by Naoki Sato.*

Solution 1 by: Sam Elder (11/CO)

If n is even, then

$$S(n) = (1 - 2) + (3 - 4) + \cdots + [(n - 1) - n] = \underbrace{-1 - 1 - \cdots - 1}_{n/2\ -1\text{s}} = -\frac{n}{2}.$$

If n is odd, then $S(n) = S(n - 1) + n = -\frac{n-1}{2} + n = \frac{n+1}{2}$. We now consider the expression $T(m, n) = S(m) + S(n) + S(m + n)$.

Case 1. Both m and n are odd. Then $m + n$ is even, so

$$T(m, n) = \frac{m + 1}{2} + \frac{n + 1}{2} - \frac{m + n}{2} = 1.$$

Case 2. Both m and n are even. Then $m + n$ is even, so

$$T(m, n) = -\frac{m}{2} - \frac{n}{2} - \frac{m + n}{2} = -m - n < 0.$$

Case 3. m is odd and n is even. Then $m + n$ is odd, so

$$T(m, n) = \frac{m + 1}{2} - \frac{n}{2} + \frac{m + n + 1}{2} = m + 1,$$

which is even.

Case 4. n is odd and m is even. Analogously with the previous case, $T(m, n) = n + 1$, which is again even.

None of these cases yield $T(m, n) = 2007$, so there are no solutions to part (a). For part (b), we can use either case 3 or 4, with the only difference being the ordering in the pairs. In Case 3, $m = 2007$ and n is even, and in Case 4, $n = 2007$ and m is even. Hence, the solutions are $(c, d) = (2007, n)$ and $(c, d) = (n, 2007)$, where n is any even positive integer.