

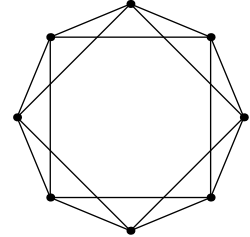


USA Mathematical Talent Search

Solutions to Problem 1/4/19

www.usamts.org

1/4/19. In the diagram at right, each vertex is labeled with a different positive factor of 2008, such that if two vertices are connected by an edge, then the label of one vertex divides the label of the other vertex. In how many different ways can the vertices be labeled? Two labelings are considered the same if one labeling can be obtained by rotating and/or reflecting the other labeling.

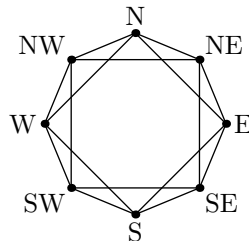


Comments We can solve the problem by using casework. The number of cases can be reduced by using the symmetry in the diagram. Starting with the factors 8 and 251 is also a good idea, because they are the “least connectable” factors. *Solutions edited by Naoki Sato.*

Solution by: Sam Elder (12/CO)

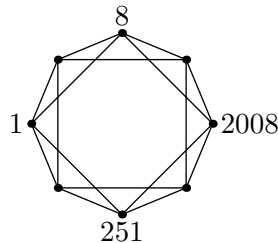
The positive factors of 2008 are 1, 2, 4, 8, 251, 502, 1004, and 2008. Six pairs of these numbers do not satisfy the given property (one divides the other): (2, 251), (4, 251), (4, 502), (8, 251), (8, 502), and (8, 1004). These pairs cannot be connected by an edge.

Denote the vertices as N, NW, W, SW, S, SE, E, and NE as directions on a compass.



Rotate any working grid so 251 is at S. Each vertex in this diagram is connected by edges to four others, and not connected to the other three. Since 251 cannot be connected to 2, 4, or 8, these three numbers must be in the northernmost three places in the diagram, N, NE and NW. Consider the different positions 8 can be in.

Case 1: 8 is at N. The only two factors that can lie at W and E are 1 and 2008. Reflect any diagrams across the north-south axis so 1 is at W and 2008 is at E.



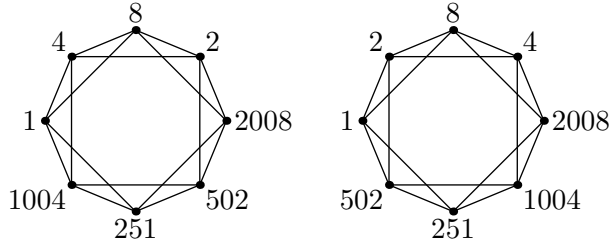
2 and 4 occupy NE and NW, so 502 and 1004 occupy SE and SW. The only other forbidden pair is (4, 502), so if 4 is at NE then 502 is at SW, and if 4 is at NW then 502 is at SE. These are the two solutions for this case:



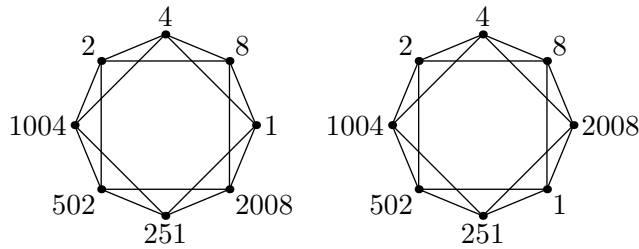
USA Mathematical Talent Search

Solutions to Problem 1/4/19

www.usamts.org



Case 2: 8 is not at N. Reflect across the north-south axis so 8 is at NE. Then 2 and 4 are at N and NW, while 502 and 1004 are at W and SW to avoid edges with 8. The only way 4 and 502 cannot share an edge is if 4 is at N and 502 at SW. Then 2 is at NW and 1004 at W, and the only factors left to arrange are 1 and 2008, which can go either at E or SE:



With two possible labelings in each case, there are 4 possible labelings in all.