

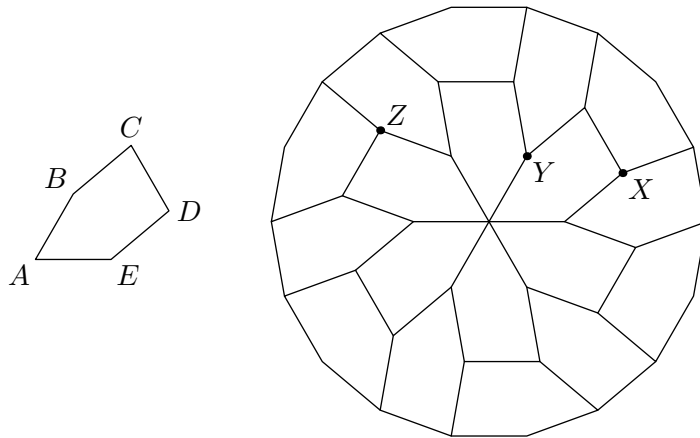


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Solutions to Problem 2/1/19

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2/1/19. A regular 18-gon is dissected into 18 pentagons, each of which is congruent to pentagon $ABCDE$, as shown. All sides of the pentagon have the same length.



- (a) Determine angles A , B , C , D , and E .
- (b) Show that points X , Y , and Z are collinear.

Credit This problem was proposed by Naoki Sato.

Comments Part (a) can be done by considering appropriate combinations of angles in the regular 18-gon. Part (b) can be done by showing that $\angle XYZ = 180^\circ$. *Solutions edited by Naoki Sato.*

Solution 1 by: Luyi Zhang (9/CT)

(a) At the center of the 18-gon, six pentagons join together by their angle that corresponds to $\angle A$. Therefore, $\angle A = 360^\circ/6 = 60^\circ$. Since all sides of the pentagon are equal, triangle ABE is equilateral and quadrilateral $BCDE$ is a rhombus.

$\angle ABC$ is an interior angle of the 18-gon, so $\angle B = \angle ABC = 160^\circ$. Then

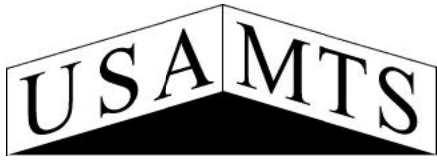
$$\angle EBC = \angle ABC - \angle ABE = 160^\circ - 60^\circ = 100^\circ,$$

so $\angle D = \angle CDE = \angle EBC = 100^\circ$ and

$$\angle C = \angle BED = 180^\circ - \angle EBC = 180^\circ - 100^\circ = 80^\circ.$$

Finally, $\angle E = \angle AED = \angle AEB + \angle BED = 60^\circ + 80^\circ = 140^\circ$.

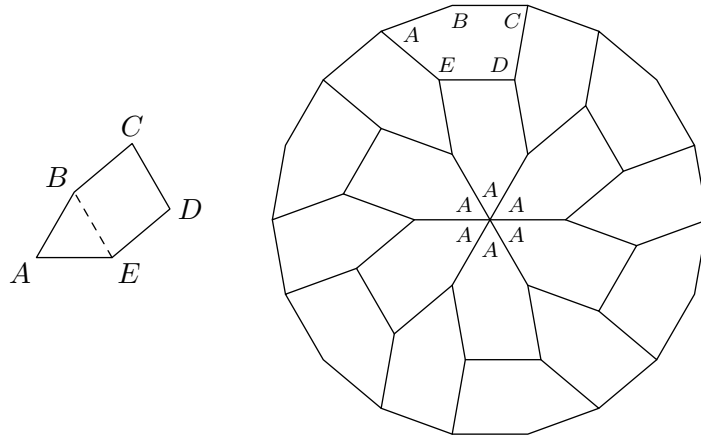
To summarize, $\angle A = 60^\circ$, $\angle B = 160^\circ$, $\angle C = 80^\circ$, $\angle D = 100^\circ$, and $\angle E = 140^\circ$.



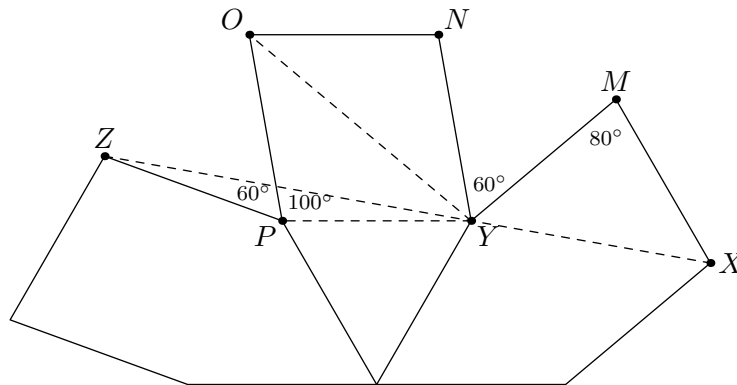
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(b) To show that points X , Y , and Z are collinear we will show that $\angle XYZ = 180^\circ$. Label points M , N , O , and P , as shown below.



Since all the sides are of equal length, we can easily create isosceles triangles to assist in our angle search. In triangle MYX , $MX = MY$ and $\angle XMY = 80^\circ$, so $\angle MYX = \angle MYX = (180^\circ - 80^\circ)/2 = 50^\circ$.

In triangle PYZ , $PY = PZ$ and $\angle ZPY = \angle ZPO + \angle OPY = 60^\circ + 100^\circ = 160^\circ$, so $\angle PZY = \angle PYZ = (180^\circ - 160^\circ)/2 = 10^\circ$.

Then in triangle OPY , $PO = PY$ and $\angle OPY = 100^\circ$, so $\angle PYO = \angle POY = \angle NYO = \angle NOY = (180^\circ - 100^\circ)/2 = 40^\circ$, so $\angle ZYO = \angle PYO - \angle PYZ = 40^\circ - 10^\circ = 30^\circ$. Then

$$\angle XYZ = \angle MYX + \angle MYN + \angle NYO + \angle ZYO = 50^\circ + 60^\circ + 40^\circ + 30^\circ = 180^\circ,$$

and we are done.