
2/3/17. Anna writes a sequence of integers starting with the number 12. Each subsequent integer she writes is chosen randomly with equal chance from among the positive divisors of the previous integer (including the possibility of the integer itself). She keeps writing integers until she writes the integer 1 for the first time, and then she stops. One such sequence is

$$12, 6, 6, 3, 3, 3, 1.$$

What is the expected value of the number of terms in Anna's sequence?

Credit This problem was proposed by Mathew Crawford.

Comments This problem is similar to problem 2/1/17, in which we also calculated an expected value using a recursive formula. *Solutions edited by Naoki Sato.*

Solution 1 by: Garrett Marcotte (12/CA)

Let (a_n) be a sequence such as described in the problem, and let $E(a_1)$ be the expected number of terms of (a_n). To calculate $E(a_1)$, suppose that d_1, d_2, $\dots$, d_k are the positive divisors of a_1, with $d_k = a_1$. Then there is a $\frac{1}{k}$ probability that any given divisor d_i is chosen as a_2. Thus, based on the method of generating the sequence, we can calculate $E(a_1)$ as follows:

$$\begin{aligned} E(a_1) &= \frac{1}{k}[E(d_1) + 1] + \frac{1}{k}[E(d_2) + 1] + \cdots + \frac{1}{k}[E(d_k) + 1] \\ &= \frac{1}{k}[k + E(d_1) + E(d_2) + \cdots + E(d_{k-1})] + \frac{1}{k}E(d_k) \\ \Rightarrow \quad \frac{k-1}{k}E(a_1) &= \frac{1}{k}[k + E(d_1) + E(d_2) + \cdots + E(d_{k-1})] \\ \Rightarrow \quad E(a_1) &= \frac{1}{k-1}[k + E(d_1) + E(d_2) + \cdots + E(d_{k-1})]. \end{aligned}$$

Now we apply this result to find $E(12)$. By the definition of the sequence, $E(1) = 1$. The numbers 2 and 3 have the same number of divisors, namely 2, so

$$E(2) = E(3) = \frac{1}{1}[2 + E(1)] = 3.$$

The number 4 has three divisors, namely 1, 2, and 4, so

$$E(4) = \frac{1}{2}[3 + E(1) + E(2)] = \frac{7}{2}.$$

The number 6 has four divisors, namely 1, 2, 3, and 6, so

$$E(6) = \frac{1}{3}[4 + E(1) + E(2) + E(3)] = \frac{11}{3}.$$

Finall, the number 12 has six divisors, namely 1, 2, 3, 4, 6, and 12, so

$$\begin{aligned} E(12) &= \frac{1}{5}[6 + E(1) + E(2) + E(3) + E(4) + E(6)] \\ &= \frac{1}{5}\left(6 + 1 + 3 + 3 + \frac{7}{2} + \frac{11}{3}\right) \\ &= \frac{121}{30}. \end{aligned}$$

Solution 2 by: Gaku Liu (10/FL)

We will count the expected value of the number of each of the integers 12, 6, 4, 3, 2, and 1 in the sequence separately. (Note: We will use the term *decomposition* to denote a term changing from one integer to a different one.)

The integer 12 always appears as the first term of the sequence. The next integer has an equal chance of being any one of 12's six divisors, so a second 12 will appear an expected $\frac{1}{6}$ times. Then, a third 12 will appear an expected $(\frac{1}{6})^2$ times, etc., so the expected value of the number of 12's is

$$1 + \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \cdots = \frac{6}{5}.$$

The integer 6 can only decompose from the integer 12. 12 has an equal chance of decomposing into any of its five proper divisors, so 6 has a $\frac{1}{5}$ chance of appearing in the sequence. 6 has four divisors, so the expected value of the number of 6's is

$$\frac{1}{5}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] = \frac{1}{5} \cdot \frac{4}{3} = \frac{4}{15}.$$

The integer 4 can only decompose from the integer 12, so it has a $\frac{1}{5}$ chance of appearing in the sequence. 4 has three divisors, so the expected value of the number of 4's is

$$\frac{1}{5}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right] = \frac{1}{5} \cdot \frac{3}{2} = \frac{3}{10}.$$

The integer 3 can decompose from either 12 or 6. It has a $\frac{1}{5}$ chance of decomposing from 12. 6 has a $\frac{1}{5}$ chance of appearing in the sequence, and has three proper divisors it can decompose into, so there is a $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$ chance the integer 3 will decompose from 6. Hence, there is a $\frac{1}{5} + \frac{1}{15} = \frac{4}{15}$ chance 3 will appear in the sequence. The integer 3 has two divisors, so the expected value of the number of 3's is

$$\frac{4}{15}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{4}{15} \cdot 2 = \frac{8}{15}.$$

The integer 2 can decompose from either 12, 6, or 4. It has a $\frac{1}{5}$ chance of decomposing from 12, a $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$ chance of decomposing from 6, and a $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$ chance of decomposing from 4. Hence, 2 has a $\frac{1}{5} + \frac{1}{15} + \frac{1}{10} = \frac{11}{30}$ chance of appearing in the sequence. The integer 2 has two divisors, so the expected value of the number of 2's is

$$\frac{11}{30}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{11}{30} \cdot 2 = \frac{11}{15}.$$

The integer 1 will always appear exactly 1 time. Hence, the expected value of the total number of terms is

$$\frac{6}{5} + \frac{4}{15} + \frac{3}{10} + \frac{8}{15} + \frac{11}{15} + 1 = \frac{121}{30}.$$