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Solutions to Problem 2/3/18

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2/3/18. Find, with proof, all real numbers x between 0 and 2π such that

$$\tan 7x - \sin 6x = \cos 4x - \cot 7x.$$

Credit This problem was proposed by Marcin E. Kuczma from the University of Warsaw.

Comments By expressing everything in terms of sine and cosine, it is seen that the given equation is actually an inequality in disguise, where equality must occur. The equation can then be solved by finding where equality occurs. *Solutions edited by Naoki Sato.*

Solution 1 by: Bohua Zhan (12/NJ)

Writing everything in terms of sine and cosine and rearranging, we have:

$$\begin{aligned}
&\frac{\sin 7x}{\cos 7x} - \sin 6x = \cos 4x - \frac{\cos 7x}{\sin 7x} \\
\Leftrightarrow \quad &\frac{\sin 7x}{\cos 7x} + \frac{\cos 7x}{\sin 7x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad &\frac{\sin^2 7x + \cos^2 7x}{\sin 7x \cos 7x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad &\frac{1}{\sin 7x \cos 7x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad &\frac{2}{\sin 14x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad &2 = \sin 14x(\cos 4x + \sin 6x).
\end{aligned}$$

Since the range of sine and cosine are $[-1, 1]$, $|\sin 14x| \le 1$ and $|\cos 4x + \sin 6x| \le 2$ for all x. Since the product of these two expressions is 2, they must all attain the maximum value. That is, $|\sin 14x| = 1$, $|\sin 6x| = 1$, and $\cos 4x = \sin 6x$. There are two cases:

Case 1: If $\sin 14x = -1$, then $\cos 4x = \sin 6x = -1$. So $4x = k\pi$, where k is an odd integer. Then for x between 0 and 2π, we have $x = \frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$. It is not difficult to verify that only $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ satisfy the other two equations.

Case 2: If $\sin 14x = 1$, then $\cos 4x = \sin 6x = 1$. So $4x = k\pi$, where k is an even integer. For x between 0 and 2π, we have $x = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π. Note that for all four possible values of x, $6x$ is a multiple of π, and $\sin 6x = 0$. Therefore, there are no solutions in this case.

In conclusion, the solutions of x between 0 and 2π are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.