2/4/16. Find positive integers a, b, and c such that

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}}.$$

Prove that your solution is correct. (Warning: numerical approximations of the values do not constitute a proof.)

Credit This is based on Problem 80 on page 42 of *Problems from the History of Mathematics*, by Lévárdi and Sain, a book published in Hungarian in Budapest, 1982. Problem 80 was attributed to the Indian mathematician Bhaskara (1114 - ca. 1185).

Comments Most students squared both sides of the given equation to get the solution. Examples are given below by Jason Ferguson and Tony Liu. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Jason Ferguson (12/TX)

If the ordered triple of real numbers (a, b, c) satisfy the problem condition, then so will any permutation (a', b', c') of (a, b, c). Thus, we may assume without loss of generality that $a \le b \le c$.

Upon squaring both sides of the equation

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}},$$

we obtain

$$\begin{aligned} a+b+c+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc} &= 219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280} \\ &= 219+12\sqrt{70}+30\sqrt{14}+84\sqrt{5}. \end{aligned}$$

As a, b, and c are integers with $a \le b \le c$, it can be the case that $a+b+c=219$, $2\sqrt{ab}=12\sqrt{70}$, $2\sqrt{ac}=30\sqrt{14}$, and $2\sqrt{bc}=84\sqrt{5}$. Then

$$\begin{aligned} \sqrt{ab} &= 6\sqrt{70}, &(1)\\ \sqrt{ac} &= 15\sqrt{14}, &(2)\\ \sqrt{bc} &= 42\sqrt{5}. &(3) \end{aligned}$$

Multiplying (1), (2), and (3) gives

$$abc = 264600, \qquad (4)$$

and squaring (1), (2), and (3) gives

$$\begin{aligned} ab &= 2520, &(5)\\ ac &= 3150, &(6)\\ bc &= 8820, &(7) \end{aligned}$$

respectively. Dividing (4) by (7), (6), and (5), respectively give $a = 30$, $b = 84$, and $c = 105$. Then indeed $a + b + c = 219$, so $\boxed{(a, b, c) = (30, 84, 105)}$ satisfy

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{219 + \sqrt{10080} + \sqrt{12600} + \sqrt{35280}},$$

as desired. QED

Solution 2 by: Tony Liu (10/IL)
Squaring the given equation, we obtain

$$a + b + c + 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca} = 219 + \sqrt{10080} + \sqrt{12600} + \sqrt{35280}.$$

Since there are three radical terms on the right side (which are not integers), the three radicals on the left side can match up correspondingly. Also note that because a, b, c are positive integers, this implies $a + b + c = 219$. Without loss of generality, we may assume $b \le a \le c \Rightarrow ab \le bc \le ca$ to obtain the following system of equations:

$$\begin{aligned}
\sqrt{ab} = \frac{1}{2}\sqrt{10080} &= 6\sqrt{70} \\
\sqrt{bc} = \frac{1}{2}\sqrt{12600} &= 15\sqrt{14} \\
\sqrt{ca} = \frac{1}{2}\sqrt{35280} &= 42\sqrt{5}
\end{aligned}$$

Multiplying the three gives $abc = 264600 \Rightarrow \sqrt{abc} = 210\sqrt{6}$. Thus, we can solve for a, b, c:

$$\sqrt{a} = \frac{\sqrt{abc}}{\sqrt{bc}} = \frac{210\sqrt{6}}{15\sqrt{14}} = 2\sqrt{21} \Rightarrow a = 84.$$

$$\sqrt{b} = \frac{\sqrt{abc}}{\sqrt{ca}} = \frac{210\sqrt{6}}{42\sqrt{5}} = \sqrt{30} \Rightarrow b = 30.$$

$$\sqrt{c} = \frac{\sqrt{abc}}{\sqrt{ab}} = \frac{210\sqrt{6}}{6\sqrt{70}} = \sqrt{105} \Rightarrow c = 105.$$

Checking, we note that $a + b + c = 84 + 30 + 105 = 219$ still holds. Finally, we conclude that $a = 84, b = 30, c = 105$ satisfy the given equation.