

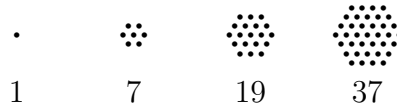


# USA Mathematical Talent Search

Solutions to Problem 2/4/17

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**2/4/17.** *Centered hexagonal numbers* are the numbers of dots used to create hexagonal arrays of dots. The first four centered hexagonal numbers are 1, 7, 19, and 37, as shown below.



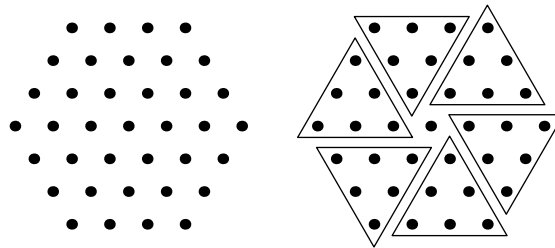
Centered Hexagonal Numbers

Consider an arithmetic sequence  $1, a, b$  and a geometric sequence  $1, c, d$ , where  $a, b, c$ , and  $d$  are all positive integers and  $a + b = c + d$ . Prove that each centered hexagonal number is a possible value of  $a$ , and prove that each possible value of  $a$  is a centered hexagonal number.

**Credit** This problem was proposed by Richard Rusczyk and Erin Schram.

**Comments** This problem requires some algebra to find the  $n^{\text{th}}$  centered hexagonal number, and then a little number theory to show the equivalence. *Solutions edited by Naoki Sato.*

**Solution 1 by: Mike Nasti (11/IL)**



We want to find an explicit formula for the  $n^{\text{th}}$  centered hexagonal number. Partitioning the dots as above, we see immediately that the  $n^{\text{th}}$  centered hexagonal number is 1 more than 6 times the  $(n - 1)^{\text{th}}$  triangular number. Thus, the  $n^{\text{th}}$  centered hexagonal number is  $1 + 6 \cdot \frac{(n-1)(n)}{2} = 3n^2 - 3n + 1$ .

The arithmetic sequence  $1, a, b$  has common difference  $a - 1$ , so it can be written in one variable as  $1, a, 2a - 1$ , so  $b = 2a - 1$ . The geometric sequence  $1, c, d$  has common ratio  $c$ , so it too can be written in one variable as  $1, c, c^2$ , so  $d = c^2$ . Then  $a + b = c + d \Rightarrow a + 2a - 1 = c + c^2 \Rightarrow 3a - 1 = c(c + 1)$ .

Let  $c = 3n - 2$  for some integer  $n$ . Then  $c$  is an integer, so we know  $3a - 1 = (3n - 2)(3n - 2 + 1) = 9n^2 - 9n + 2 = 3(3n^2 - 3n + 1) - 1$ . Thus, whenever  $c = 3n - 2$ ,  $a = 3n^2 - 3n + 1$  which is the  $n^{\text{th}}$  centered hexagonal number. So each centered hexagonal number is a possible value of  $a$ .



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Now, if  $c(c+1) = 3a - 1$  for an integer  $a$ , we can say  $c(c+1) \equiv -1 \equiv 2 \pmod{3}$ . Then we know that  $c \equiv 1 \pmod{3}$  because if  $c \equiv 0 \pmod{3}$ , then  $c(c+1) \equiv 0 \pmod{3}$ , and if  $c \equiv 2 \pmod{3}$ , then  $c(c+1) \equiv 0 \pmod{3}$ . Since  $c \equiv 1 \equiv -2 \pmod{3}$ , every possible value of  $c$  can be written in the form  $3n - 2$  for some integer  $n$ . Therefore the set of possible values of  $a$  is equal to the set of centered hexagonal numbers, so every possible value of  $a$  is a centered hexagonal number.