USA Mathematical Talent Search

Solutions to Problem 2/4/17

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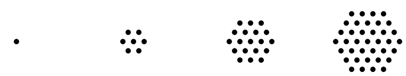
2/4/17. *Centered hexagonal numbers* are the numbers of dots used to create hexagonal arrays of dots. The first four centered hexagonal numbers are 1, 7, 19, and 37, as shown below.



1 7 19 37

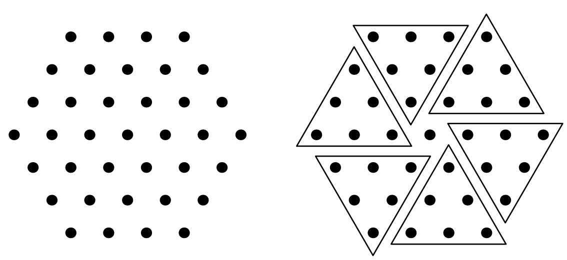
Centered Hexagonal Numbers

Consider an arithmetic sequence 1, a, b and a geometric sequence 1, c, d, where a, b, c, and d are all positive integers and $a+b=c+d$. Prove that each centered hexagonal number is a possible value of a, and prove that each possible value of a is a centered hexagonal number.

Credit This problem was proposed by Richard Rusczyk and Erin Schram.

Comments This problem requires some algebra to find the n^{th} centered hexagonal number, and then a little number theory to show the equivalence. *Solutions edited by Naoki Sato.*

Solution 1 by: Mike Nasti (11/IL)



We want to find an explicit formula for the n^{th} centered hexagonal number. Partitioning the dots as above, we see immediately that the n^{th} centered hexagonal number is 1 more than 6 times the $(n-1)^{\text{th}}$ triangular number. Thus, the n^{th} centered hexagonal number is $1+6\cdot\frac{(n-1)(n)}{2}=3n^2-3n+1$.

The arithmetic sequence 1, a, b has common difference $a-1$, so it can be written in one variable as 1, a, $2a-1$, so $b=2a-1$. The geometric sequence 1, c, d has common ratio c, so it too can be written in one variable as 1, c, c^2, so $d=c^2$. Then $a+b=c+d \Rightarrow a+2a-1=c+c^2 \Rightarrow 3a-1=c(c+1)$.

Let $c=3n-2$ for some integer n. Then c is an integer, so we know $3a-1=(3n-2)(3n-2+1)=9n^2-9n+2=3(3n^2-3n+1)-1$. Thus, whenever $c=3n-2$, $a=3n^2-3n+1$ which is the n^{th} centered hexagonal number. So each centered hexagonal number is a possible value of a.

Now, if $c(c+1) = 3a-1$ for an integer a, we can say $c(c+1) \equiv -1 \equiv 2 \pmod 3$. Then we know that $c \equiv 1 \pmod 3$ because if $c \equiv 0 \pmod 3$, then $c(c+1) \equiv 0 \pmod 3$, and if $c \equiv 2 \pmod 3$, then $c(c+1) \equiv 0 \pmod 3$. Since $c \equiv 1 \equiv -2 \pmod 3$, every possible value of c can be written in the form $3n-2$ for some integer n. Therefore the set of possible values of a is equal to the set of centered hexagonal numbers, so every possible value of a is a centered hexagonal number.