



## USA Mathematical Talent Search

Solutions to Problem 3/2/16

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**3/2/16.** A set is *reciprocally whole* if its elements are distinct integers greater than 1 and the sum of the reciprocals of all those elements is exactly 1. Find a set  $S$ , as small as possible, that contains two reciprocally whole subsets,  $I$  and  $J$ , which are distinct but not necessarily disjoint (meaning they may share elements, but they may not be the same subset). Prove that no set with fewer elements than  $S$  can contain two reciprocally whole subsets.

**Credit** We are thankful to Dr. Kent D. Boklan of the National Security Agency for devising this nice problem.

**Comments** There are many possible 5-element sets which satisfy the conditions of the problem; probably the one most commonly cited in student's solutions was  $\{2, 3, 4, 6, 12\}$ . The key to this problem was *rigorously* proving that a set with 4 or fewer elements is impossible. Solution 1 is an especially concise example. Some solutions, such as Solution 2, prove along the way that  $\{2, 3, 6\}$  is the unique reciprocally whole set with 3 elements.

### Solution 1 by: Zhou Fan (11/NJ)

A reciprocally whole set must have at least three elements, since the reciprocals of only two distinct integers greater than 1 can sum to at most  $1/2 + 1/3 = 5/6$ . Thus a set  $S$  with two reciprocally whole subsets must contain at least four elements. Suppose such a set exists:  $S = \{a, b, c, d\}$ . The entire four element set cannot be reciprocally whole if there is a smaller reciprocally whole subset, so  $S$  must contain two three-element reciprocally whole subsets. WLOG, assume that they are  $\{a, b, c\}$  and  $\{a, b, d\}$ . Then  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$  and  $\frac{1}{a} + \frac{1}{b} + \frac{1}{d} = 1$ , which implies that  $\frac{1}{c} = \frac{1}{d}$ , or  $c = d$ : a contradiction. Thus  $S$  has at least 5 elements. Such a five-element set exists:  $S = \{2, 3, 6, 7, 42\}$  satisfies the problem conditions since  $1/2 + 1/3 + 1/6 = 1$  and  $1/2 + 1/3 + 1/7 + 1/42 = 1$ .

### Solution 2 by: Jason Ferguson (12/TX)

Consider the set  $S = \{2, 3, 6, 9, 18\}$ .  $S$  is a 5-element set that contains two distinct reciprocally whole subsets,  $\{2, 3, 6\}$  and  $\{2, 3, 9, 18\}$  (since  $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = 1$ ). We will now show that there are no sets with two distinct reciprocally whole subsets with cardinality 0, 1, 2, 3, or 4.

To this end, we first show that there cannot be a reciprocally whole set of cardinality zero, one, or two and the only reciprocally whole set of cardinality three is  $\{2, 3, 6\}$ . Clearly the null set cannot be reciprocally whole, and if the reciprocal of a number is 1, then that number must be one. Thus, there is only 1 one-element set  $S$  which has the property that the sum of the reciprocals of its elements, and that set is  $\{1\}$ , but this is not a reciprocally whole set (all elements must be greater than 1). Suppose  $T$  is a reciprocally whole set of cardinality two, and let  $x$  be the smaller element and  $y$  the larger (the two elements are distinct). Then  $\{x, y\}$  is a reciprocally whole set, so  $\frac{1}{x} + \frac{1}{y} = 1$ . However, because  $x < y$ ,  $\frac{1}{x} > \frac{1}{y}$ , so  $\frac{1}{x} > \frac{1}{2}$ . Then  $x < 2$ . But all elements in a reciprocally whole set must be integers greater than one.



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From this contradiction we conclude that there are no 2-element reciprocally whole sets.

Suppose now that  $U$  is a three-element reciprocally whole set. Then, let  $a$  be the smallest element,  $b$  the middle element, and  $c$  the largest. Then  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ , and  $a < b < c$ . Therefore,  $\frac{1}{a} > \frac{1}{b} > \frac{1}{c}$ , so  $\frac{1}{a} > \frac{1}{3}$ . Then  $a < 3$ , so  $a = 2$ . Because  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$  and  $a = 2$ , it follows that  $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$ . Because  $\frac{1}{b} > \frac{1}{c}$ , it follows that  $\frac{1}{b} > \frac{1}{4}$ . Then  $b < 4$ . Since  $2 = a < b$ , it follows that  $b = 3$ . Because  $\frac{1}{b} + \frac{1}{c} = \frac{1}{2}$  and  $b = 3$ , it follows that  $\frac{1}{c} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ . Then,  $c = 6$ , so  $(a, b, c) = (2, 3, 6)$ , and we conclude that the only reciprocally whole three-element set is  $\{2, 3, 6\}$ . Therefore, there is only one reciprocally whole set with cardinality less than or equal to 3:  $\{2, 3, 6\}$ .

A set with cardinality of 0 or 1 cannot have two distinct, nonempty subsets. Since a reciprocally whole set has to be nonempty, it follows that a set of cardinality 0 or 1 cannot have two distinct reciprocally whole subsets. Now, if a set  $V$  has two distinct reciprocally whole subsets, then neither of those two subsets can be  $V$  itself, for if  $V$  was reciprocally whole, then the other reciprocally whole subset of  $V$  must be a proper subset of  $V$ . Then the sum of the reciprocals of the elements of the proper subset would be less than the sum of the reciprocals of the elements of  $V$ , which is 1. From this contradiction we conclude that if a set  $V$  has two reciprocally whole subsets, then both of them must be proper subsets of  $V$ . Thus, if a set with cardinality 2 had two distinct, reciprocally whole subsets, then both of them would have to have cardinality less than or equal to 1. This is impossible, as there are no reciprocally whole sets of cardinality 0 or 1. Similarly, because there are also no reciprocally whole sets of cardinality 2, a set with cardinality 3 cannot have two distinct reciprocally whole subsets. Finally, if a set with cardinality 4 had two distinct, reciprocally whole subsets, then both of them would have to have cardinality less than or equal to 3. This is impossible, as there is only one reciprocally whole set of cardinality less than or equal to 3.

So we conclude that there are no sets with two distinct, reciprocally whole subsets whose cardinality is less than or equal to 4, but the set  $S = \{2, 3, 6, 9, 18\}$  is a five-element set with this property. We also conclude that  $S$  is a minimal set with two distinct, reciprocally whole subsets. QED