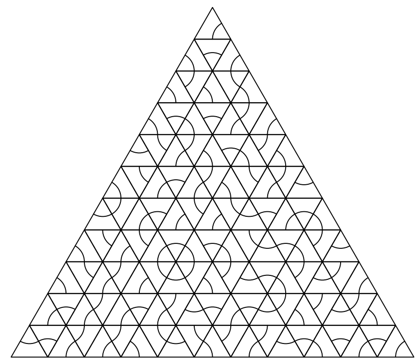
3/2/17. An equilateral triangle is tiled with n^2 smaller congruent equilateral triangles such that there are n smaller triangles along each of the sides of the original triangle. The case $n = 11$ is shown at right. For each of the small equilateral triangles, we randomly choose a vertex V of the triangle and draw an arc with that vertex as center connecting the midpoints of the two sides of the small triangle with V as an endpoint. Find, with proof, the expected value of the number of full circles formed, in terms of n.



Credit This problem was proposed by Richard Rusczyk.

Comments Trying to count in how many cases the circles appear gets very complicated, as these cases are not independent. (In other words, whether a circle appears at one vertex affects whether a circle can appear at adjacent vertices.) However, a fundamental property of expected value is that the expected value of a sum is simply the sum of the expected values, a property mentioned in our Expected Value article. Once you see this, the problem actually becomes quite easy. *Solutions edited by Naoki Sato.*

Solution 1 by: Derrick Sund (12/WA)

Consider a vertex in such a triangle that has six small triangles around it. Each of these triangles has a $\frac{1}{3}$ probability of its arc being the arc of a circle centered on that vertex. Therefore, the probability that that vertex has a full circle around it is $\frac{1}{3^6} = \frac{1}{729}$. To get the expected value of the number of full circles, we simply need to multiply this probability by the number of vertices that can have full circles around them. If the original triangle is divided into n^2 smaller triangles, then the number of such vertices will be

$$(n-2) + (n-3) + (n-4) + \cdots + 2 + 1 = \frac{(n-1)(n-2)}{2},$$

so the desired expected value is

$$\frac{(n-1)(n-2)}{2} \cdot \frac{1}{729} = \frac{(n-1)(n-2)}{1458}.$$