



# USA Mathematical Talent Search

Solutions to Problem 3/2/19

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**3/2/19.** A triangular array of positive integers is called *remarkable* if all of its entries are distinct, and each entry, other than those in the top row, is the quotient of the two numbers immediately above it. For example, the following triangular array is remarkable:

$$\begin{array}{ccccc} & & 7 & & 42 & & 14 & & \\ & & & & 6 & & & & 3 & & \\ & & & & & & & & & & 2 & & \end{array}$$

Find the smallest positive integer that can occur as the greatest element in a remarkable array with four numbers in the top row.

**Comments** It is fairly easy to find an example where the greatest number in the array is 120. Then, one can prove that 120 is the minimum by showing that some number in the top row is the product of four different integers, all at least 2. *Solutions edited by Naoki Sato.*

**Solution by: Dmitri Gekhtman (11/IN)**

A remarkable array cannot contain a 1, because this would mean that it contains at least two equal numbers. Denote the integer in the bottom row by  $a_1$ . Then the second row from the bottom contains integers  $a_2$  and  $a_1a_2$ , with  $a_2 \neq a_1$ . In the third row from the bottom, and above  $a_1a_2$ , there are integers  $a_3$  and  $a_1a_2a_3$  (in either order), with  $a_3$  different from  $a_1$  and  $a_2$ . Finally, in the top row, and above  $a_1a_2a_3$ , there are integers  $a_4$  and  $a_1a_2a_3a_4$  (in either order), with  $a_4$  different from  $a_1$ ,  $a_2$ , and  $a_3$ .

This means that the greatest number in the top row is at least  $a_1a_2a_3a_4$ . Since  $a_1a_2a_3a_4$  is a product of four different integers, all at least 2, it is greater or equal than  $2 \cdot 3 \cdot 4 \cdot 5 = 120$ . Therefore, the answer to the problem is at least 120. An example with the greatest element in the array equal to 120 is as follows:

$$\begin{array}{ccccccc} & & & & 40 & & 5 & & 120 & & 30 & & \\ & & & & & & 8 & & 24 & & & & 4 & & \\ & & & & & & & & 3 & & & & 6 & & \\ & & & & & & & & & & & & & & 2 & \end{array}$$

Hence, the answer is 120.