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Solutions to Problem 3/2/19
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3/2/19. A triangular array of positive integers is called *remarkable* if all of its entries are distinct, and each entry, other than those in the top row, is the quotient of the two numbers immediately above it. For example, the following triangular array is remarkable:

$$\begin{array}{ccccc} 7 & & 42 & & 14 \\ & 6 & & 3 & \\ & & 2 & & \end{array}$$

Find the smallest positive integer that can occur as the greatest element in a remarkable array with four numbers in the top row.

Comments It is fairly easy to find an example where the greatest number in the array is 120. Then, one can prove that 120 is the mininum by showing that some number in the top row is the product of four different integers, all at least 2. *Solutions edited by Naoki Sato.*

Solution by: Dmitri Gekhtman (11/IN)

A remarkable array cannot contain a 1, because this would mean that it contains at least two equal numbers. Denote the integer in the bottom row by a_1. Then the second row from the bottom contains integers a_2 and a_1a_2, with $a_2 \neq a_1$. In the third row from the bottom, and above a_1a_2, there are integers a_3 and $a_1a_2a_3$ (in either order), with a_3 different from a_1 and a_2. Finally, in the top row, and above $a_1a_2a_3$, there are integers a_4 and $a_1a_2a_3a_4$ (in either order), with a_4 different from a_1, a_2, and a_3.

This means that the greatest number in the top row is at least $a_1a_2a_3a_4$. Since $a_1a_2a_3a_4$ is a product of four different integers, all at least 2, it is greater or equal than $2 \cdot 3 \cdot 4 \cdot 5 = 120$. Therefore, the answer to the problem is at least 120. An example with the greatest element in the array equal to 120 is as follows:

$$\begin{array}{ccccccc} 40 & & 5 & & 120 & & 30 \\ & 8 & & 24 & & 4 & \\ & & 3 & & 6 & & \\ & & & 2 & & & \end{array}$$

Hence, the answer is 120.