
3/3/16. Define the recursive sequence 1, 4, 13, … by $s_1 = 1$ and $s_{n+1} = 3s_n + 1$ for all positive integers n. The element $s_{18} = 193710244$ ends in two identical digits. Prove that all the elements in the sequence that end in two or more identical digits come in groups of three consecutive elements that have the same number of identical digits at the end.

Credit This problem was devised by Erin Schram of the NSA. It is based on an "Olympiad Problem" of a 2003 issue of the *Gazeta Matematica* magazine that was posted on the Art of Problem Solving forum.

Comments Many students proved that the last two digits repeat in a cycle of 20, and used this cycle to prove that the elements in the sequence that end in two or more identical digits come in groups of three consecutive elements. Fewer students proved the second half – that within each of these groups of three, the three numbers have the identical number of repeating digits at the end. Jeffrey Manning gives a clear, concise explanation, and Cary Malkiewich gives us a more formal solution. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Jeffrey Manning (9/CA)

If a number ends in two or more identical digits its last two digits must be identical. Working out the sequence modulo 100 gives:

$$1, 4, 13, 40, 21, 64, 93, 80, 41, 24, 73, 20, 61, 84, 53, 60, 81, \mathbf{44}, \mathbf{33}, \mathbf{00}, 1, \ldots$$

Since the sequence is recursive and $s_{21} \equiv s_1 \equiv 1 \bmod 100$ the sequence modulo 100 must repeat every 20 elements which means that all elements that end in two or more identical digits come in groups of three consecutive elements, where the digits are 4s in the first element, 3s in the second and 0s in the third. Now we must prove that they end in the same number of identical digits.

Let n be the number of 4s a the end of some element of the sequence. Since,

$$3(\underbrace{444\ldots4}_{n\ digits}) + 1 = 1\underbrace{333\ldots3}_{n\ digits} \quad \text{and} \quad 3(\underbrace{333\ldots3}_{n\ digits}) + 1 = 1\underbrace{000\ldots0}_{n\ digits}$$

each element must end in at least as many identical digits as the previous element.

For the second element to end in more than n identical digits the last $n+1$ digits of the first element must be $X\underbrace{444\ldots4}_{n\ digits}$, where X is a digit other than 4 such that $3X + 1 \equiv 3 \bmod 10$, but the only single digit that would satisfy this is 4 which is a contradiction. This means that the second element must end in exactly n identical digits.

Similarly, for the third element to end in more than n digits the last $n+1$ digits of the second element must be $Y\underbrace{333\ldots3}_{n\ digits}$ where $Y \neq 3$ and $3Y+1 \equiv 0 \mod 10$, but similarly the only single digit that would satisfy this is 3 which is a contradiction. So all three elements must end in exactly n elements. The proof is complete.

Solution 2 by: Cary Malkiewich (12/MA)

Define $\varphi_0 : \mathbf{Z}_{10} \to \mathbf{Z}_{10}$ and $\varphi_1 : \mathbf{Z}_{10} \to \mathbf{Z}_{10}$ as follows:

$$\begin{aligned}\varphi_0(a) &= 3a \mod 10\\ \varphi_1(a) &= 3a+1 \mod 10\end{aligned}$$

Lemma: The functions φ_0 and φ_1 are bijective.
Proof: This is proven simply by listing out elements.

$\varphi_0(0)=0$	$\varphi_0(5)=5$	$\varphi_1(0)=1$	$\varphi_1(5)=6$
$\varphi_0(1)=3$	$\varphi_0(6)=8$	$\varphi_1(1)=4$	$\varphi_1(6)=9$
$\varphi_0(2)=6$	$\varphi_0(7)=1$	$\varphi_1(2)=7$	$\varphi_1(7)=2$
$\varphi_0(3)=9$	$\varphi_0(8)=4$	$\varphi_1(3)=0$	$\varphi_1(8)=5$
$\varphi_0(4)=2$	$\varphi_0(9)=7$	$\varphi_1(4)=3$	$\varphi_1(9)=8$

Since every element of $\mathbf{Z}_{10}$ appears exactly once in the range of each function, each function is bijective. ■

As a result of this lemma, we can define φ_0^{-1} and φ_1^{-1} to be the inverses of the above functions.

Since $\varphi_1(1)=4$, $\varphi_1(4)=3$, $\varphi_1(3)=0$, and $\varphi_1(0)=1$, the units digit in the given sequence cycles through 1,4,3,0. These are only 4 numbers that could form the repeating digits at the end of s_n.

In order to rigorously prove the assertion, we must prove all three of these statements ($k>1$):
1. Iff s_n ends in exactly k 4's, s_{n+1} ends in exactly k 3's.
2. Iff s_{n+1} ends in exactly k 3's, s_{n+2} ends in exactly k 0's.
3. s_n can never end in two or more 1's.

1. Iff s_n ends in exactly k 4's, s_{n+1} ends in exactly k 3's.
Assume a term s_n of the sequence ends in a string of 4's that is exactly k digits long: $(k > 1)$

$$\ldots x444\ldots444$$

When s_n is multiplied by 3, every 4 becomes a 12. A 1 is carried in every column, resulting in a string of $(k-1)$ 3's followed by a 2:

$$\ldots[\varphi_1(x)]333\ldots332$$

Then 1 is added, and the new number $3s_n + 1 = s_{n+1}$ ends in a string of 3's that is k digits long:

$$\ldots[\varphi_1(x)]333\ldots333$$

The $(k+1)$ digit from the right, $\varphi_1(x)$, is not a 3. Since φ_1 is bijective, this would imply that the digit x in s_n is $\varphi_1^{-1}(3) = 4$, and we have assumed that only the last k digits were 4.

For the converse, suppose that s_{n+1} ends in exactly k 3's. s_n ends in a 4, so suppose s_n ends with exactly m 4's. By the argument above, s_{n+1} ends in exactly m 3's. Therefore $m = k$, and s_n ends with exactly k 4's. ■

2. Iff s_{n+1} ends in exactly k 3's, s_{n+2} ends in exactly k 0's.
Suppose s_{n+1} ends in a string of 3's that is exactly k digits long: $(k > 1)$

$$\ldots y333\ldots333$$

Then $3s_{n+1}$ will end in a string of 9's that is exactly k digits long:

$$\ldots[\varphi_0(y)]999\ldots999$$

$3s_{n+1} + 1 = s_{n+2}$ will then end in a string of 0's that is k digits long:

$$\ldots[\varphi_1(y)]000\ldots000$$

The $(k+1)$ digit from the right, $\varphi_1(y)$, is not a 0. This would imply that the digit y in s_{n+1} is $\varphi_1^{-1}(0) = 3$, and we have assumed that only the last k digits were 3.

For the converse, suppose that s_{n+2} ends in exactly k 0's. s_{n+1} ends in a 3, so suppose s_{n+1} ends with exactly m 3's. By the argument above, s_{n+2} ends in exactly m 0's. Therefore $m = k$, and s_{n+1} ends with exactly k 3's. ■

3. s_n can never end in two or more 1's.
The last two digits form this repeating sequence of 20 terms:

$$01, 04, 13, 40, 21, 64, 93, 80, 41, 24, 73, 20, 61, 84, 53, 60, 81, 44, 33, 00$$

Since 11 is not a member of this sequence, no member of the original sequence s_n can end in two or more 1's. ■

Now that all three statements have been proven, it follows that all elements in the sequence that end in two or more identical digits (4, 3, 0) come in groups of three consecutive elements $(44\ldots44, 33\ldots33, 00\ldots00)$ that have the same number (k) of identical digits at the end.