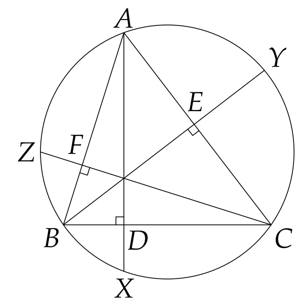
3/3/17. Points A, B, and C are on a circle such that $\triangle ABC$ is an acute triangle. X, Y, and Z are on the circle such that AX is perpendicular to BC at D, BY is perpendicular to AC at E, and CZ is perpendicular to AB at F. Find the value of

$$\frac{AX}{AD}+\frac{BY}{BE}+\frac{CZ}{CF},$$

and prove that this value is the same for all possible A, B, C on the circle such that $\triangle ABC$ is acute.



Credit This problem was proposed by Naoki Sato.

Comments This geometry problem can be solved by recognizing that the given ratios can be expressed as ratios of certain areas, and using the fundamental result that $HD = DX$, where H is the orthocenter of triangle ABC. A solution using power of a point is also possible. *Solutions edited by Naoki Sato.*

Solution 1 by: Justin Hsu (11/CA)

Let H be the orthocenter of $\triangle ABC$. First, $\triangle BHD$ is similar to $\triangle BCE$, since they are both right triangles and they share $\angle CBE$, so $\angle BCE = \angle BHD$. Also, $\angle BXA = \angle BCA = \angle BHD$, since they both are inscribed angles that intercept the same arc BA. Now, $\triangle BXH$ is isosceles, which means that BD is the perpendicular bisector of segment HX. Therefore, $\triangle BDH \equiv \triangle BDX$, and $HD = DX$. Similarily, this can be extended to the other sides of the triangle to show that $HE = EY$ and $HF = FZ$.

Now,

$$\begin{aligned}\frac{AX}{AD}+\frac{BY}{BE}+\frac{CZ}{CF} &= \frac{AD+DX}{AD}+\frac{BE+EY}{BE}+\frac{CF+FZ}{CF}\\ &= 1+\frac{DX}{AX}+1+\frac{EY}{BY}+1+\frac{FZ}{CZ}\\ &= 3+\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF}.\end{aligned}$$

But each fraction is a ratio between the altitudes of two triangles with the same base, so we can rewrite this sum in terms of area, where $[ABC]$ denotes the area of $\triangle ABC$:

$$\begin{aligned}3+\frac{HD}{AD}+\frac{HE}{BE}+\frac{HF}{CF} &= 3+\frac{[HBC]}{[ABC]}+\frac{[HCA]}{[ABC]}+\frac{[HAB]}{[ABC]}\\ &= 3+\frac{[ABC]}{[ABC]}\\ &= 3+1=4.\end{aligned}$$

Solution 2 by: James Sundstrom (11/NJ)

By the Power of a Point Theorem,

$$AD \cdot DX = BD \cdot CD.$$

Therefore,

$$\frac{DX}{AD} = \frac{BD}{AD} \cdot \frac{CD}{AD} = \cot B \cot C.$$

Similarly,

$$\frac{EY}{BE} = \cot C \cot A,$$
$$\frac{FZ}{CF} = \cot A \cot B.$$

We can calculate

$$\begin{aligned}\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} &= \frac{AD}{AD} + \frac{DX}{AD} + \frac{BE}{BE} + \frac{EY}{BE} + \frac{CF}{CF} + \frac{FZ}{CF} \\ &= 3 + \frac{DX}{AD} + \frac{EY}{BE} + \frac{FZ}{CF} \\ &= 3 + \cot B \cot C + \cot C \cot A + \cot A \cot B \\ &= 3 + \frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C}.\end{aligned}$$

We claim that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

for all acute triangles $\triangle ABC$ (acuteness of $\triangle ABC$ means that $\tan A$, $\tan B$, and $\tan C$ exist). [Ed: As the following argument shows, this identity holds for all triangles ABC where both sides are defined.]

We have that

$$\tan C = \tan(\pi - A - B) = \tan(-A - B) = -\tan(A + B),$$

and

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

so

$$\tan C = -\frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

This can be re-arranged to become $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Hence,

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} = 3 + 1 = 4.$$