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Solutions to Problem 3/4/18
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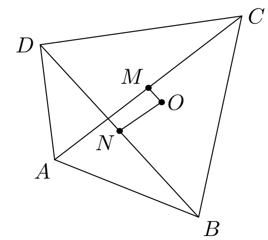
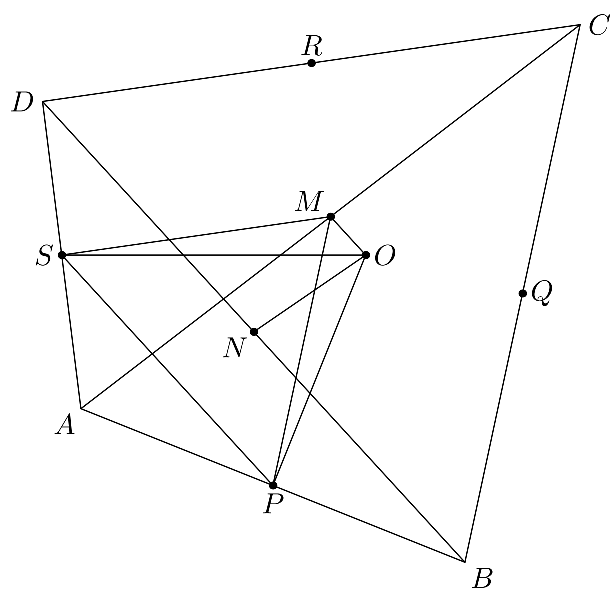
3/4/18. Let $ABCD$ be a convex quadrilateral. Let M be the midpoint of diagonal $\overline{AC}$ and N be the midpoint of diagonal $\overline{BD}$. Let O be the intersection of the line through N parallel to $\overline{AC}$ and the line through M parallel to $\overline{BD}$. Prove that the line segments joining O to the midpoints of each side of $ABCD$ divide $ABCD$ into four pieces of equal area.



Credit This problem is based on a problem from the Canadian IMO training program.

Comments This problem can be succinctly solved by using formulas for the areas of triangles involving their bases and heights. *Solutions edited by Naoki Sato.*

Solution 1 by: Kenan Diab (12/OH)



Let P, Q, R, and S be the midpoints of AB, BC, CD, and DA, respectively. Consider quadrilateral $APMS$. By definition of midpoint, we have $AB = 2AP$, $AC = 2AM$, and $AD = 2AS$. Thus, a homothecy centered at A maps quadrilateral $APMS$ to quadrilateral $ABCD$ with a factor of 2. Hence, $[APMS] = [ABCD]/4$ and $SP \parallel BD$.

But, we are given $OM \parallel BD$, so $OM \parallel SP$. Thus, O and M are the same distance from SP. Since $\triangle OSP$ and $\triangle MSP$ share side SP, it follows that $[OSP] = [MSP]$. Thus,

$$[OPAS] = [OSP] + [ASP] = [MSP] + [ASP] = [APMS] = \frac{[ABCD]}{4}.$$

Analogous homothecies centered at B, C, and D give $[OPAS] = [OQBP] = [ORCQ] = [OSDR] = [ABCD]/4$, as desired.