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Solutions to Problem 4/1/17
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4/1/17. Homer gives mathematicians Patty and Selma each a different integer, not known to the other or to you. Homer tells them, within each other's hearing, that the number given to Patty is the product ab of the positive integers a and b, and that the number given to Selma is the sum $a+b$ of the same numbers a and b, where $b > a > 1$. He doesn't, however, tell Patty or Selma the numbers a and b. The following (honest) conversation then takes place:

Patty: "I can't tell what numbers a and b are."
Selma: "I knew before that you couldn't tell."
Patty: "In that case, I now know what a and b are."
Selma: "Now I also know what a and b are."

Supposing that Homer tells *you* (but neither Patty nor Selma) that neither a nor b is greater than 20, find a and b, and prove your answer can result in the conversation above.

Credit This problem comes from the Carnegie Mellon Math Studies Problem Seminar.

Comments What makes this problem tricky is that it's not just a problem: It's a problem inside a problem. It requires you to place yourselves in the shoes of Patty and Selma, and not only make the same deductions they make, but deduce which conditions could have led to those deductions. Some careful casework leads to the answer. Note that in the following solutions, Meir Lakhovsky shows that $(a, b) = (4, 13)$ is a viable solution, and Jeffrey Manning shows that it is the only solution. *Solutions edited by Naoki Sato.*

Solution 1 by: Meir Lakhovsky (10/WA)

Some trial and error leads us to $a = 4$, $b = 13$. Let us show that the conversation mentioned could take place. Patty was given the number 52 and Selma was given the number 17.

Patty said, "I can't tell what numbers a and b are," which is true since the product 52 could be achieved through either the numbers (2,26) or (4,13).

Selma answers "I knew before that you couldn't tell," which is also true, since for each possible pair (a, b) that adds up to 17, namely (2,15), (3,14), (4,13), (5,12), (6,11), (7,10), and (8,9), there are at least two possible solutions that give rise to the product ab.

Patty then says, "In that case, I now know what a and b are," which is true because if (a, b) was equal to (2,26), then Selma would have had the number 28, which means she could not have made her former statement because (a, b) could have been (5,23) in which case Patty would have been able to figure out what a and b are before any statements were made. Since Patty knows the product is 52 and (a, b) is not (2,26), she knows that (a, b) is (4,13), the only other option.

Selma now says, "Now I also know what a and b are," which is true since the only pair from (2,15), (3,14), (4,13), (5,12), (6,11), (7,10), and (8,9) in which Patty could have made her later statement is (4,13) (reasoning shown above). Let us show why (2,15), (3,14), (5,12), (6,11), (7,10), and (8,9) don't work. If (2,15) were the numbers, then Patty would have had the number 30 and would not have been able to make her later statement since both (2,15) and (5,6) yield sums for which Selma would have been able to make her former statement. Likewise, for (3,14), Patty would not have been able to make her statement since both (3,14) and (2,21) yield sums for which Selma would have been able to make her former statement. By the same logic, the pair for (5,12) is (3,20); the pair for (6,11) is (2,33); the pair for (7,10) is (2,35); and the pair for (8,9) is (3,24).

Thus, $(a, b) = (4, 13)$ could have resulted in the described conversation.

Solution 2 by: Jeffrey Manning (10/CA)

Notice that Patty could tell what a and b were if and only if there is exactly one way to factor ab into the product of two distinct integers, both greater than one (for the rest of the solution we will use the word factorization to mean factorization into two distinct factors both greater than 1), in that case a and b are the two factors. This would only happen when $ab = pq$ where p and q are primes, in which case a and b would be p and q or when $ab = p^3$ where p is prime, in which case $a = p^2$ and $b = p$.

For Selma to already know that Patty couldn't tell what a and b were, it must be impossible to write $a + b$ as the sum of two distinct primes or as the sum of a prime and its square. Since $a + b \leq 40$, the possible values for $a + b$ are 11, 17, 23, 27, 29, 35, and 37 (notice that we do need to consider primes and squares of primes greater than 20 because *Selma* doesn't know that $a, b \leq 20$).

For the third line of the conversation to be true, there must be exactly one factorization of ab such that the sum of the factors cannot be written as the sum of two distinct primes or as the sum of a prime and its square. If $a, b > 1$, then $(a - 1)(b - 1) = ab - (a + b) + 1 > 0$, so $a + b < ab + 1$, so Patty knows that $a + b < 401$. This means Patty knows that $a + b$ is either odd or twice a prime (Goldbach's conjecture states that any even integer ≥ 4 is the sum of two primes, not necessarily distinct. Although this has not been proven, it has been verified for all values we are concerned with). So we only need to consider factorizations of ab such that the sum of the factors is not divisible by 4 (we can't have $a + b = 2(2) = 4$ because that would mean ab is 3 or 4). Notice that if $ab = 4p$ where p is an odd prime, then the only possible values of a and b are 4 and p.

For the fourth line to be true there must be only one way to write $a + b$ as the sum of two numbers (> 1) whose product satisfies these conditions.

We have:

If $a+b=11=4+7=2+9$, then we could have $ab=28$ or 18.
If $a+b=23=4+19=7+16$, then we could have $ab=76$ or 112.
If $a+b=27=4+23=2+25$, then we could have $ab=92$ or 50.
If $a+b=29=13+16=2+27$, then we could have $ab=208$ or 54.
If $a+b=35=4+31=32+3$, then we could have $ab=124$ or 96.
If $a+b=37=8+29=32+5$, then we could have $ab=232$ or 160.

We will show that each of these possible values of ab satisfy the conditions for the third line to be true. The numbers 28, 76, 92 and 124 are all in the form $4p$ so they all work.

If $ab=18=2\cdot 3\cdot 3$ then the only factorization, other than $2\cdot 9$, is $3\cdot 6$ which gives $a+b=9=2+7$.

If $ab=50=2\cdot 5\cdot 5$ then the only factorization, other than $2\cdot 25$, is $5\cdot 10$ which gives $a+b=15=2+13$.

If $ab=54=2\cdot 3\cdot 3\cdot 3$ then the only factorizations, other than $2\cdot 27$, are $6\cdot 9$ which gives $a+b=15=2+13$, and $3\cdot 18$ which gives $a+b=21=2+19$.

If $ab=112=2\cdot 2\cdot 2\cdot 2\cdot 7$ then the only factorizations other than $7\cdot 16$ such that $4\nmid a+b$, are $8\cdot 14$ which gives $a+b=22=5+17$, and $2\cdot 56$ which gives $a+b=58=5+53$.

If $ab=208=2\cdot 2\cdot 2\cdot 2\cdot 13$ then the only factorizations other than $13\cdot 16$ such that $4\nmid a+b$, are $8\cdot 26$ which gives $a+b=34=3+31$, and $2\cdot 104$ which gives $a+b=106=47+59$.

If $ab=96=2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 3$ then the only factorizations other than $3\cdot 32$ such that $4\nmid a+b$, are $6\cdot 16$ which gives $a+b=22=5+17$, and $2\cdot 48$ which gives $a+b=50=19+31$.

If $ab=160=2\cdot 2\cdot 2\cdot 2\cdot 2\cdot 5$ then the only factorizations other than $5\cdot 32$ such that $4\nmid a+b$, are $10\cdot 16$ which gives $a+b=26=3+23$, and $2\cdot 80$ which gives $a+b=82=29+53$.

If $ab=232=2\cdot 2\cdot 2\cdot 29$ then the only factorizations, other than $8\cdot 29$, are $4\cdot 58$ which gives $a+b=62=3+59$, and $2\cdot 116$ which gives $a+b=118=5+113$.

Therefore $a+b=11$, 23, 27, 29, 35, and 37 don't satisfy the fourth line, so $a+b=17$.

Thus the possible ordered pairs (a,b) are:

$(2,15)\Rightarrow ab=30$, but this can be factored as $5\cdot 6$ and $5+6=11$.
$(3,14)\Rightarrow ab=42$, but this can be factored as $2\cdot 21$ and $2+21=23$.
$(4,13)\Rightarrow ab=52$, which can be written in the form $4p$ so it satisfies the third line.
$(5,12)\Rightarrow ab=60$, but this can be factored as $3\cdot 20$ and $3+20=23$.
$(6,11)\Rightarrow ab=66$, but this can be factored as $2\cdot 33$ and $2+33=35$.
$(7,10)\Rightarrow ab=70$, but this can be factored as $2\cdot 35$ and $2+35=37$.
$(8,9)\Rightarrow ab=72$, but this can be factored as $3\cdot 24$ and $3+24=27$.

Thus the only value of ab that satisfies the third line is 52, so $a + b = 17$ satisfies the fourth line. Thus the only possible value of (a, b) is (4,13).