



USA Mathematical Talent Search

Solutions to Problem 4/3/18

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4/3/18. Alice plays in a tournament in which every player plays a game against every other player exactly once. In each game, either one player wins and earns 2 points while the other gets 0 points, or the two players tie and both players earn 1 point. After the tournament, Alice tells Bob how many points she earned. Bob was not in the tournament, and does not know what happened in any individual game of the tournament.

- (a) Suppose there are 12 players in the tournament, including Alice. What is the smallest number of points Alice could have earned such that Bob can deduce that Alice scored more points than at least 8 other players?
- (b) Suppose there are n players in the tournament, including Alice, and that Alice scored m points. Find, in terms of n and k , the smallest value of m such that Bob can deduce that Alice scored more points than at least k other players.

Credit This problem is based on a problem that appeared in *Problem Solving Journal for secondary students*, a British publication.

Comments To show that the answer is $m = n + k - 1$, we must prove two things: First, we must prove that it is possible for Alice to score $n + k - 2$ points while beating at most $k - 1$ other players. Second, we must prove that if Alice scores $n + k - 1$ points, then she must have scored more points than at least k other players. *Solutions edited by Naoki Sato.*

Solution 1 by: Matt Superdock (10/PA)

We will find a generalization in part (b) and use it to find the answer to part (a).

We claim that the smallest value of m such that Bob can deduce that Alice scored more points than at least k other players is $m = n + k - 1$.

If Alice scores more points than at least k other players, then there are at most $n - k$ players, including Alice, that scored m or more points. It is sufficient to prove that it is impossible for $n - k + 1$ players to each score $n + k - 1$ or more points, and that it is possible for $n - k + 1$ players to each score $n + k - 2$ or more points.

Suppose for the sake of contradiction that $n - k + 1$ players each score $n + k - 1$ or more points. Call these $n - k + 1$ players *good* players, and call the other $k - 1$ players *bad* players. The good players must collectively score a total of at least $(n - k + 1)(n + k - 1)$ points.

There are $\binom{n-k+1}{2}$ games between two good players, and there are two points available in each game. In these games, good players can accumulate a total of $2\binom{n-k+1}{2} = (n - k + 1)(n - k)$ points. There are $(n - k + 1)(k - 1)$ games between one good player and one bad player, and there are again two points available in each game. In these games, good players



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can accumulate a total of $(n - k + 1)(2k - 2)$ points. Good players cannot accumulate points in games between two bad players. Therefore, good players can accumulate a total of at most $(n - k + 1)(n - k) + (n - k + 1)(2k - 2) = (n - k + 1)(n + k - 2)$ points, which is less than $(n - k + 1)(n + k - 1)$. We have reached a contradiction, so it is impossible for $n - k + 1$ players to each score $n + k - 1$ or more points.

It remains to be shown that it is possible for $n - k + 1$ players to each score $n + k - 2$ or more points. Again, call these $n - k + 1$ players *good* players, and call the other $k - 1$ players *bad* players. Every good player plays $n - k$ games against good players and $k - 1$ games against bad players. Suppose that every game between two good players results in a tie, and every game between a good player and a bad player results in a win for the good player. In this scenario, each good player gets exactly $n - k + 2(k - 1) = n + k - 2$ points, as desired.

Therefore the answer is indeed $m = n + k - 1$.

Applying our generalization to part (a), we find that the answer is $12 + 8 - 1 = 19$.

Further Comments. Some students attempted to prove that if Alice scores $n + k - 1$ points, then she must have scored more points than at least k other players by setting up a “worst-case scenario,” consisting of a number of top players that tie each other, and beat all worse players.

Proofs that simply state this worst-case scenario are not sufficiently rigorous. Proofs that began with this scenario and used the idea that for every point one player gained, another player lost, were also generally unconvincing. You must prove that Alice scores more points than at least k other players in all possible scenarios, and the easiest way is as done above.