USAMTS

USA Mathematical Talent Search

Solutions to Problem 5/1/16

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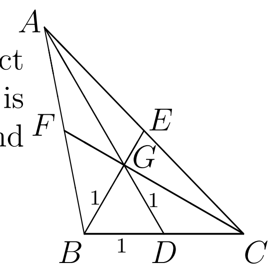


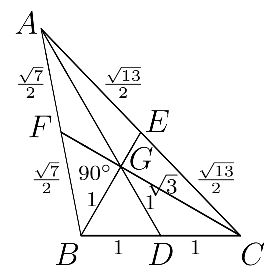
5/1/16. Point G is where the medians of the triangle ABC intersect and point D is the midpoint of side $\overline{BC}$. The triangle BDG is equilateral with side length l. Determine the lengths, AB, BC, and CA, of the sides of triangle ABC.

Credit The problem was proposed by Professor Gregory Galperin, a member of the Committee in charge of the USA Mathematical Olympiad. He has suggested many excellent problems to the USAMTS over the years.

Comments Our participants found numerous solutions to this problem. Below are 4 of the most common approaches. Other solutions involved complex numbers, clever rotations or line extensions, and complicated analytic geometry. Our first solution below, from Timothy Zhu, exhibits the basic geometric approach. The second solution, from David Benjamin, employs the law of cosines, while Lawrence Chan gives us a third solution using analytic geometry. Solution 4 from Ameya Velingker uses Stewart's Theorem to prove a formula relating median lengths to the side lengths of a triangle, and Solution 5 from Joshua Horowitz uses vectors and analytic geometry.

Solution 1 by: Timothy Zhu (12/NH)
Lemma 1: $\overline{BE} \perp \overline{CF}$
Proof: Since D is the midpoint of $\overline{BC}$, $\overline{DC}$ has side length 1. Since $\triangle BGD$ is equilateral, its angles are all $60°$. Therefore, $\angle GDC = 180° - 60° = 120°$. Since $\overline{DG} = \overline{DC}$, $\triangle GDC$ is isosceles. Thus, $\angle DGC$ and $\angle DCG$ are both $\frac{180° - \angle GDC}{2} = 30°$, so $\angle BGC = \angle BGD + \angle DGC = 90°$. Therefore, $\overline{BE} \perp \overline{CF}$.

——end Lemma 1——



Since $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians, D, E, and F are midpoints. Therefore,

$$\overline{AB} = 2\overline{FB}$$
$$\overline{BC} = 2$$
$$\overline{CA} = 2\overline{EC}$$

Since $\overline{BE} \perp \overline{CF}$ (Lemma 1), $\overline{FB}$, $\overline{EC}$, and $\overline{GC}$ can be calculated using the Pythagorean Theorem.

$$\overline{AB} = 2\overline{FB} = 2\sqrt{\overline{BG}^2 + \overline{GF}^2}$$
$$\overline{BC} = 2$$
$$\overline{CA} = 2\overline{EC} = 2\sqrt{\overline{GC}^2 + \overline{GE}^2}$$
$$\overline{GC} = \sqrt{\overline{BC}^2 - \overline{BG}^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$$

It is well known that the centroid, G, divides the medians in 2:1 ratios. Thus,

$$\overline{GF} = \frac{\overline{GC}}{2} \text{ and } \overline{GE} = \frac{\overline{BG}}{2}$$

So,

$$\overline{AB} = 2\overline{FB} = 2\sqrt{\overline{BG}^2 + \overline{GF}^2} = 2\sqrt{\overline{BG}^2 + \left(\frac{\overline{GC}}{2}\right)^2} = 2\sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{7}$$

$$\overline{BC} = 2$$

$$\overline{CA} = 2\overline{EC} = 2\sqrt{\overline{GC}^2 + \overline{GE}^2} = 2\sqrt{\overline{GC}^2 + \left(\frac{\overline{BG}}{2}\right)^2} = 2\sqrt{\sqrt{3}^2 + \left(\frac{1}{2}\right)^2} = \sqrt{13}$$

Solution 2 by: David Benjamin (8/IN)

Side BC

D is the midpoint of $\overline{BC}$, so $DC = 1$, and $BC = 2$.

Side AB

G is the centroid of the triangle, so $\frac{AG}{GD} = 2$, so $AG = 2$, so $AD = 3$.
$\triangle BDG$ is an equilateral triangle, so $\angle GDB = 60°$.
So, by the Law of Cosines,

$$\begin{aligned}
(AB)^2 &= (BD)^2 + (AD)^2 - 2(AD)(BD)\cos\angle GDB \\
&= 1^2 + 3^2 - 2 \times 3 \times 1 \times \cos 60° \\
&= 1 + 9 - 6 \times \frac{1}{2} \\
&= 10 - 3 \\
&= 7 \\
AB &= \sqrt{7}
\end{aligned}$$

Side AC

$\angle GDC$ and $\angle GDB$ are supplementary, so $\angle GDC = 120°$.
So, by the Law of Cosines,

$$\begin{aligned}
(AC)^2 &= (DC)^2 + (AD)^2 - 2(DC)(AD)\cos\angle GDC \\
&= 1^2 + 3^2 - 2 \times 1 \times 3 \times \cos 120° \\
&= 1 + 9 - 6 \times \left(-\frac{1}{2}\right) \\
&= 10 + 3 \\
&= 13 \\
AC &= \sqrt{13}
\end{aligned}$$

Solution 3 by: Lawrence Chan (11/IL)

We will solve this problem using a coordinate plane with B as the origin and an x-axis running parallel to $\overline{BC}$. $\overline{BD} = \overline{DC}$ because $\overline{AD}$ is a median, so the coordinates of C are twice those of D. Since $D = (1,0)$, $C = (2,0)$. $\angle GBD = 60^\circ$ and $\overline{BG} = 1$ because $\triangle BGD$ is equilateral with side length 1, so using trigonometry we get

$$G = (\cos 60^\circ, \sin 60^\circ) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

G is the centroid, since it is the intersection of the medians. Centroids have the property that their coordinates are the averages of the coordinates of the vertices. Let $A = (a_1, a_2)$, $B = (0,0)$, and $C = (2,0)$ be our three vertices. Since the centroid $G = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ has coordinates that are the averages of the respective coordinates in the vertices, we can set up two equations as follows:

$$\begin{aligned} \frac{1}{2} &= \left(\frac{a_1 + 0 + 2}{3}\right) \\ \frac{\sqrt{3}}{2} &= \left(\frac{a_2 + 0 + 0}{3}\right) \end{aligned}$$

From these we can deduce that $A = \left(-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$

Applying the distance formula to the three sides of the triangle gives us

$$\boxed{\overline{AB} = \sqrt{7},\ \overline{BC} = 2,\ \overline{CA} = \sqrt{13}}$$

■

Solution 4 by: Ameya Velingker (11/PA)

Let $m_A = AD$, $m_B = BE$, $m_C = CF$ be the lengths of the medians from A, B, and C, respectively. Also, for convenience, we let $a = BC$, $b = CA$, and $c = AB$ be the sides of the triangle. By Stewart's Theorem,

$$\begin{aligned} a\left({m_A}^2 + BD \cdot CD\right) &= b^2 \cdot BD + c^2 \cdot CD \\ a\left({m_A}^2 + \frac{a^2}{4}\right) &= b^2 \cdot \frac{a}{2} + c^2 \cdot \frac{a}{2} \\ m_A &= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \qquad (1) \end{aligned}$$

Clearly, $a = 2BD = 2$ and $m_A = 3DG = 3$ since the centroid divides each median in a $2:1$ ratio. Substituting these expressions in to (1) and simplyfying, we obtain $b^2 + c^2 = 20$. Now, we note the equation

$$m_B = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2} \qquad (2)$$

which can be derived in the same way we derived (1). Observe that $m_B = \frac{3}{2}BG = \frac{3}{2}$. Substituting this expression along with $a = 2$ into (2) and rearranging, we find that $2c^2 - b^2 = 1$. Solving this equation simultaneously with $b^2 + c^2 = 20$, we get $b = \sqrt{13}$ and $c = \sqrt{7}$. Thus, the side lengths of the triangle are 2, $\sqrt{13}$, and $\sqrt{7}$.

Solution 5 by: Joshua Horowitz (10/CT)
We apply coordinates to our diagram, so that $B = (0, 0)$, $D = (1, 0)$, and $C = (2, 0)$. BDG is equilateral, so $G = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. But also, since G is the centroid of ABC, $\vec{G} = (\vec{A} + \vec{B} + \vec{C})/3$ (using vector notation). We can use this to solve for A:

$$\begin{aligned}
\vec{G} &= \frac{\vec{A} + \vec{B} + \vec{C}}{3} \\
\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) &= \frac{\vec{A} + (0,0) + (2,0)}{3} \\
\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right) &= \vec{A} + (2,0) \\
\left(-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right) &= \vec{A}.
\end{aligned}$$

Determining side lengths is now just an application of the distance formula:

$$\begin{aligned}
AB &= \sqrt{\left(-\frac{1}{2} - 0\right)^2 + \left(\frac{3\sqrt{3}}{2} - 0\right)^2} = \sqrt{7} \\
BC &= \sqrt{(0-2)^2 + (0-0)^2} = 2 \\
CA &= \sqrt{\left(2 - \left(-\frac{1}{2}\right)\right)^2 + \left(0 - \frac{3\sqrt{3}}{2}\right)^2} = \sqrt{13}.
\end{aligned}$$