USA Mathematical Talent Search

Solutions to Problem 5/1/17

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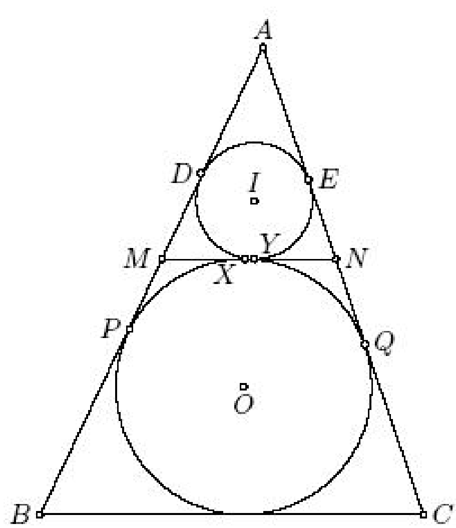
5/1/17. Given triangle ABC, let M be the midpoint of side $\overline{AB}$ and N be the midpoint of side $\overline{AC}$. A circle is inscribed inside quadrilateral $NMBC$, tangent to all four sides, and that circle touches $\overline{MN}$ at point X. The circle inscribed in triangle AMN touches $\overline{MN}$ at point Y, with Y between X and N. If $XY = 1$ and $BC = 12$, find, with proof, the lengths of the sides $\overline{AB}$ and $\overline{AC}$.

Credit This problem was proposed by Richard Rusczyk.

Comments This geometry problem is best solved using a "side chase" (as opposed to an "angle chase"), in which the relations between lengths of line segments are written down until there are a sufficient number of them that they can be solved algebraically. Any approach of this kind will almost inevitably lead to the answer. But this is not the only possible approach, as the following solutions will show. We recommend that when submitting a solution to a geometry problem to also provide a diagram, so that the grader does not have to draw one. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (11/IL)

Let AB and AC be tangent to the incircle of $\triangle AMN$ at D and E. Similarly, let AB and AC be tangent to the circle inscribed in $MNCB$ at P and Q. Let $AC = 2b$, $AB = 2c$. Note that $MN = \frac{1}{2} \cdot BC = 6$, and let $s = \frac{1}{2}(b + c + 6)$ be the semiperimeter of $\triangle AMN$.



Because $MNCB$ has an inscribed circle, $MN + BC = MB + NC$, or $b + c = 18$. By equal tangents around the incircle of $\triangle AMN$, it is easy to see that $MY = s - b$ and $NY = s - c$.

For example,

$$\begin{aligned}
2 \cdot MY &= MY + MD \\
&= (AM - AD) + (MN - YN) \\
&= AM - AE + MN - EN \\
&= AM + MN - AN \\
&= AM + MN + AN - 2AN \\
&= 2s - 2b \\
\Rightarrow MY &= s - b.
\end{aligned}$$

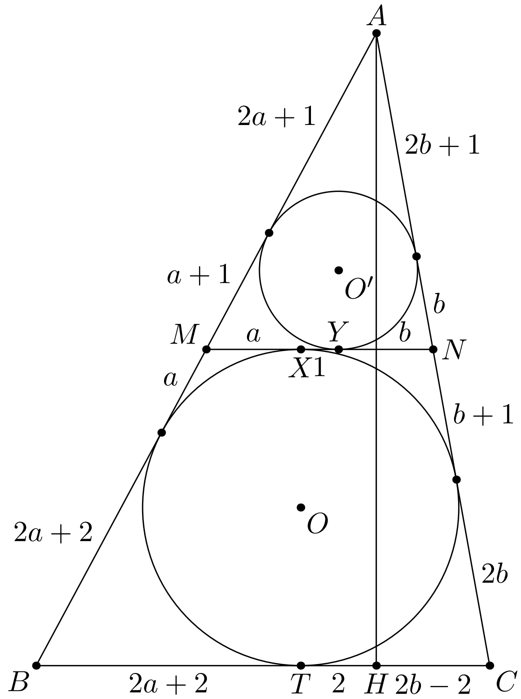
Moreover, the circle inscribed in $MNCB$ is the excircle of $\triangle AMN$, opposite of A. This implies that $NX = MY$. Indeed, we have

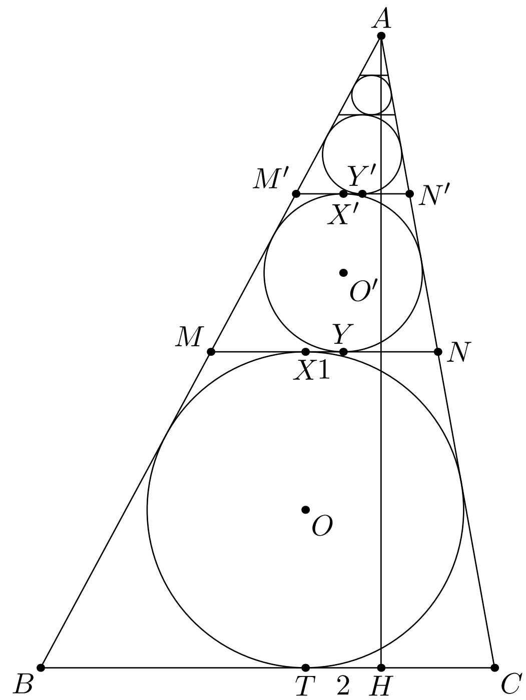
$$\begin{aligned}
2 \cdot AP &= AP + AQ \\
&= (AM + MX) + (AN + NX) \\
&= 2s \\
\Rightarrow AP &= AQ = s.
\end{aligned}$$

Thus, $MY = s - b = AQ - AN = NQ = NX$. From this, we have $1 = XY = NX - NY = (s-b)-(s-c) = c-b$. Combined with $b+c = 18$, we have $AB = 2c = 19$ and $AC = 2b = 17$.

Solution 2 by: Alan Deng (12/NY)

Let $MX = a$ and $NY = b$. By properties of tangents to a circle and the fact that triangle AMN is similar to triangle ABC with a ratio of similitude of 1:2 (since M is the midpoint of AB and N is the midpoint of AC), we can label the following diagram on the left, except for the length of TH, where H is the foot of the altitude from A to BC.





The ratio of the radius of circle O to that of O' is 2:1, since they are inscribed in similar triangles of that ratio. The distance XY is 1. Draw a line $M'N'$ parallel to MN that is also tangent to circle O', as shown in the diagram on the right. We also have points X' and Y' on $M'N'$ that correspond to points X and Y on MN. In fact, X' and Y' are the images of X and Y, respectively, of the similitude through A by ratio 1:2, so $X'Y' = XY/2 = 1/2$. Furthermore, $X'Y$ is parallel to AH since $M'N'$ and MN are parallel tangents to circle O'.

If we continue this process of drawing tangents and circles towards point A, we obtain line segments of length 1, 1/2, 1/4, 1/8, and so on, and the union of their orthogonal projections onto BC is TH. Hence,

$$TH = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

Now, we have two right triangles ABH and ACH that share the same height. Then

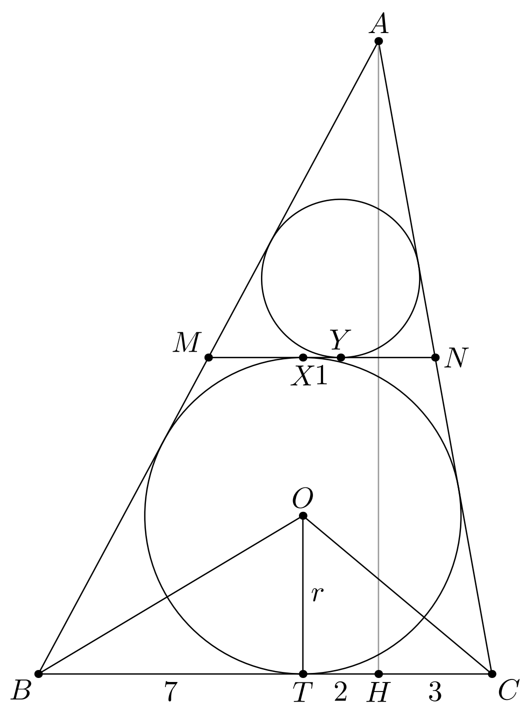
$$\begin{aligned}
AB^2 - BH^2 &= AH^2 = AC^2 - CH^2 \\
\Rightarrow\ (6a+4)^2 - (2a+4)^2 &= (6b+2)^2 - (2b-2)^2 \\
\Rightarrow\ 32a^2 + 32a &= 32b^2 + 32b \\
\Rightarrow\ 4a^2 + 4a &= 4b^2 + 4b \\
\Rightarrow\ 4a^2 + 4a + 1 &= 4b^2 + 4b + 1 \\
\Rightarrow\ (2a+1)^2 &= (2b+1)^2,
\end{aligned}$$

so $a = b$. Since $BC = 2a + 2b + 2 = 12$, $a = b = 5/2$. Hence, $AB = 6a + 4 = 19$ and $AC = 6b + 2 = 17$.

Solution 3: Based on the solution by Logan Daum (11/AK)

By the side chasing argument in Solution 1, we get that $MX = NY$. (For example, this follows immediately from $MY = NX$.) Since $XY = 1$ and $MN = 6$, we have that $MX = NY = 5/2$, and we can argue that $TH = 2$ as in Solution 2. (Actually, Logan argues this in a more direct way than Alan. Let Z be the intersection of AH and MN. What do you notice about triangles AZY and TXY?)

Now, T is the image of Y under the homothety through A by a factor of 2, so $CT = 2NY = 5$, which means $BT = BC - CT = 12 - 5 = 7$. Also, $CH = CT - TH = 3$ and $BH = BC - CH = 9$.



Let r be the inradius of triangle ABC, so the height of trapezoid $BMNC$ is $2r$, so then the height of triangle ABC is $AH = 4r$. Then

$$\tan \frac{B}{2} = \frac{OT}{BT} = \frac{r}{7},$$

and

$$\tan B = \frac{AH}{BH} = \frac{4r}{9}.$$

By the double angle formula,

$$\tan B = \frac{2\tan(B/2)}{1-\tan^2(B/2)}.$$

Substituting, we obtain

$$\begin{aligned}
\frac{4r}{9} &= \frac{2r/7}{1-(r/7)^2} = \frac{14r}{49-r^2} \\
\Rightarrow\ 196 - 4r^2 &= 126 \\
\Rightarrow\ r^2 &= \frac{70}{4} = \frac{35}{2} \\
\Rightarrow\ r &= \sqrt{\frac{35}{2}} \\
\Rightarrow \tan B &= \frac{4r}{9} = \frac{4}{9}\sqrt{\frac{35}{2}} = \frac{2\sqrt{70}}{9}.
\end{aligned}$$

Since $\tan B$ is positive, B is acute. Also,

$$\begin{aligned}
\cos^2 B &= \frac{\cos^2 B}{\cos^2 B + \sin^2 B} \\
&= \frac{1}{1+\frac{\sin^2 B}{\cos^2 B}} \\
&= \frac{1}{1+\tan^2 B} \\
&= \frac{1}{1+\frac{4\cdot 70}{81}} \\
&= \frac{81}{361},
\end{aligned}$$

so

$$\cos B = \sqrt{\frac{81}{361}} = \frac{9}{19}.$$

Hence,

$$AB = \frac{BH}{\cos B} = \frac{9}{9/19} = 19.$$

Similarly,

$$\tan C = \frac{AH}{CH} = \frac{4r}{3} = \frac{4}{3}\sqrt{\frac{35}{2}} = \frac{2\sqrt{70}}{3},$$

so C is also acute, and

$$\begin{aligned}\cos^2 C &= \frac{1}{1+\tan^2 C} \\ &= \frac{1}{1+\frac{4\cdot 70}{9}} \\ &= \frac{9}{289},\end{aligned}$$

so

$$\cos C = \sqrt{\frac{9}{289}} = \frac{3}{17}.$$

Hence,

$$AC = \frac{AH}{\cos C} = \frac{3}{3/17} = 17.$$