

USA Mathematical Talent Search

Solutions to Problem 5/1/18

www.usamts.org

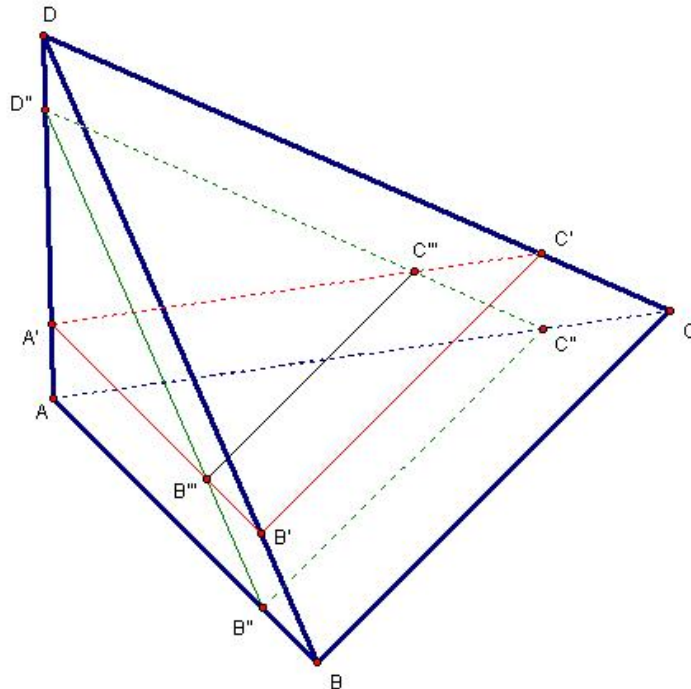
5/1/18. $ABCD$ is a tetrahedron such that $AB = 6$, $BC = 8$, $AC = AD = 10$, and $BD = CD = 12$. Plane \mathcal{P} is parallel to face ABC and divides the tetrahedron into two pieces of equal volume. Plane \mathcal{Q} is parallel to face DBC and also divides $ABCD$ into two pieces of equal volume. Line ℓ is the intersection of planes \mathcal{P} and \mathcal{Q} . Find the length of the portion of ℓ that is inside $ABCD$.

Credit This problem was proposed by Richard Rusczyk.

Comments This problem is best solved by using similar tetrahedra, and drawing a nice diagram. To solve three-dimensional geometry problems, one technique that may help is to consider the two-dimensional analogue. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (12/NJ)

Let A' denote the intersection of plane \mathcal{P} and \overline{AD} , and define points B' and C' similarly. Let B'' denote the intersection of plane \mathcal{Q} and \overline{AB} , and define points C'' and D'' similarly. Let B''' denote the intersection of \mathcal{P} , \mathcal{Q} , and face ABD , and let C''' denote the intersection of \mathcal{P} , \mathcal{Q} , and face ACD . Then the problem asks for the length of $\overline{B'''C'''}$.





USA Mathematical Talent Search

Solutions to Problem 5/1/18

www.usamts.org

Tetrahedrons $ABCD$ and $A'B'C'D$ are similar because plane \mathcal{P} is parallel to face ABC . The volume of $ABCD$ is twice the volume of $A'B'C'D$, so $A'D = AD/\sqrt[3]{2} = 10/\sqrt[3]{2}$. Similarly, $AD'' = 10/\sqrt[3]{2}$. Since $A'D + AD'' = AD + A'D''$, we find that $A'D'' = 20/\sqrt[3]{2} - 10 = 10(\sqrt[3]{4} - 1)$.

Since plane \mathcal{P} and face ABC are parallel, and plane \mathcal{Q} and face DBC are parallel, tetrahedrons $ABCD$ and $A'B''C'''D''$ are similar. Therefore,

$$B'''C''' = \frac{BC \cdot A'D''}{AD} = \frac{8 \cdot 10(\sqrt[3]{4} - 1)}{10} = 8(\sqrt[3]{4} - 1).$$