USA Mathematical Talent Search

Solutions to Problem 5/2/18

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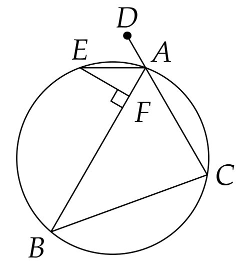
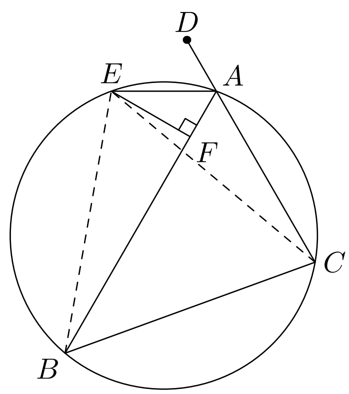
5/2/18. In triangle ABC, $AB = 8$, $BC = 7$, and $AC = 5$. We extend $\overline{AC}$ past A and mark point D on the extension, as shown. The bisector of $\angle DAB$ meets the circumcircle of $\triangle ABC$ again at E, as shown. We draw a line through E perpendicular to $\overline{AB}$. This line meets $\overline{AB}$ at point F. Find the length of $\overline{AF}$.



Credit This problem was proposed by Richard Rusczyk.

Comments An angle chase shows that triangle BEC is equilateral. Then the length of AF can be found with an application of Ptolemy's theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: Scott Kovach (11/TN)



Applying the law of cosines to triangle ABC, we see that

$$\cos \angle BAC = \frac{8^2 + 5^2 - 7^2}{2 \cdot 8 \cdot 5} = \frac{1}{2},$$

so $\angle BAC = 60^\circ$. Then $\angle EAF = \angle DAF/2 = (180^\circ - \angle BAC)/2 = 60^\circ$ as well.

Now, $\angle BEC$ subtends the same arc as $\angle BAC$, and $\angle EBC$ subtends the arc complementary to $\angle EAC$, so $\angle EBC = \angle BEC = \angle BAC = 60^\circ$, which makes triangle BEC equilateral.

Quadrilateral $EACB$ is cyclic, so by Ptolemy's theorem,

$$\begin{aligned} EA \cdot BC + EB \cdot AC &= AB \cdot EC \\ \Rightarrow \quad EA \cdot 7 + 7 \cdot 5 &= 8 \cdot 7 \\ \Rightarrow \quad EA &= 3. \end{aligned}$$

Finally, triangle EAF is a 30°-60°-90° triangle, so $AF = EA/2 = 3/2$.