5/3/18. Let $f(x)$ be a strictly increasing function defined for all $x > 0$ such that $f(x) > -\frac{1}{x}$ and $f(x)f(f(x) + \frac{1}{x}) = 1$ for all $x > 0$. Find $f(1)$.

Credit This problem was proposed by Joe Jia.

Comments One important step in solving this functional equation is to substitute $f(x)+1/x$ for x into the functional equation itself, a step which is suggested by the form of the functional equation. Then the strictly increasing condition can be used to solve for $f(x)$. *Solutions edited by Naoki Sato.*

Solution 1 by: Vlad Firoiu (9/MA)

From the given equation,

$$f\left(f(x) + \frac{1}{x}\right) = \frac{1}{f(x)}.$$

Since $y = f(x) + \frac{1}{x} > 0$ is in the domain of f, we have that

$$f\left(f(x) + \frac{1}{x}\right) \cdot f\left(f\left(f(x) + \frac{1}{x}\right) + \frac{1}{f(x) + \frac{1}{x}}\right) = 1.$$

Substituting $f\left(f(x) + \frac{1}{x}\right) = \frac{1}{f(x)}$ into the above equation yields

$$\frac{1}{f(x)} \cdot f\left(\frac{1}{f(x)} + \frac{1}{f(x) + \frac{1}{x}}\right) = 1,$$

so that

$$f\left(\frac{1}{f(x)} + \frac{1}{f(x) + \frac{1}{x}}\right) = f(x).$$

Since f is strictly increasing, it must be 1 to 1. In other words, if $f(a) = f(b)$, then $a = b$. Applying this to the above equation gives

$$\frac{1}{f(x)} + \frac{1}{f(x) + \frac{1}{x}} = x.$$

Solving yields that

$$f(x) = \frac{1 \pm \sqrt{5}}{2x}.$$

Now, if for some x in the domain of f,

$$f(x) = \frac{1 + \sqrt{5}}{2x},$$

then
$$f(x+1) = \frac{1 \pm \sqrt{5}}{2x+2} < \frac{1+\sqrt{5}}{2x} = f(x).$$
This contradicts the strictly increasing nature of f, since $x < x+1$. Therefore,
$$f(x) = \frac{1-\sqrt{5}}{2x}$$
for all $x > 0$. Plugging in $x = 1$ yields
$$f(1) = \frac{1-\sqrt{5}}{2}.$$