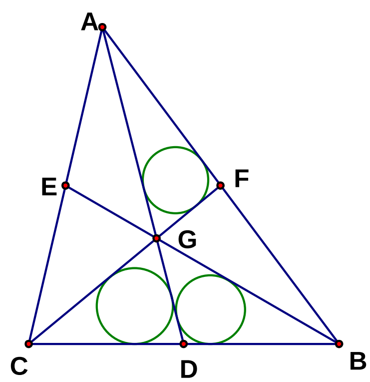
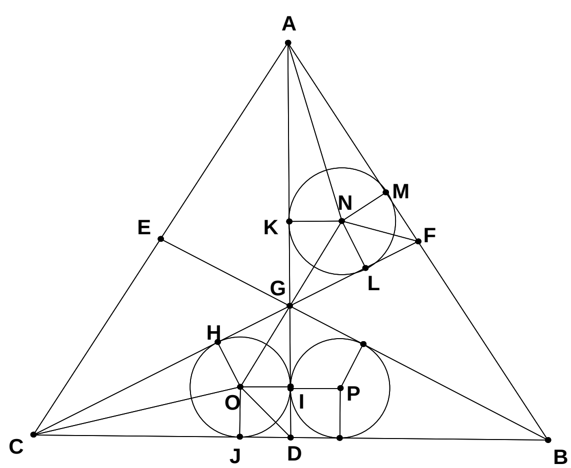
5/4/16. Medians AD, BE, and CF of triangle ABC meet at G as shown. Six small triangles, each with a vertex at G, are formed. We draw the circles inscribed in triangles AFG, BDG, and CDG as shown. Prove that if these three circles are all congruent, then ABC is equilateral.



Credit This problem was contributed by Professor Gregory Galperin, a long-time contributor of problems to the USAMTS.

Comments Most solutions involved first showing that $\triangle CGD \cong \triangle BGD$ by first showing that the perimeters of these triangles are equal. Students took a variety of approaches to showing $AF = CD$, some using a purely geometric approach, as Benjamin Dozier illustrates, some using a more trigonometric approach like that of Shotaro Makisumi, and some using a little mix of the two, like Dan Li does. *Solutions edited by Richard Rusczyk*

Solution 1 by: Benjamin Dozier (9/NM)



The area of $\triangle CDG$ equals the area of $\triangle BDG$ as they share the altitude from G to $\overline{BC}$ and they have bases of equal length. $\triangle CDG$ can be dissected into $\triangle OCD$, $\triangle ODG$ and $\triangle COG$. Likewise, $\triangle BDG$ can be dissected into $\triangle PBD$, $\triangle PDG$ and $\triangle BPG$. The area of $\triangle OCD$ equals the area of $\triangle PBD$ because they share a base and the respective altitudes to that base are of the same length. Likewise, the area of $\triangle ODG$ equals the area of $\triangle PDG$. Thus the area of $\triangle COG$ equals the area of $\triangle BPG$. Since these two triangles have altitudes of the same length, they must have bases of the same length. Therefore $CG = BG$. We know $CD = DB$, so $\triangle CDG \cong \triangle BDG$ by Side-Side-Side congruence. Furthermore, $m\angle GDC + m\angle GDB = 180°$ and so $m\angle GDC = m\angle GDB = 90°$. Since median $\overline{AD}$ is also the altitude, we know that $\triangle ABC$ is isosceles with $AC = AB$.

Now, since O and N lie on the angle bisectors of $\angle CGD$ and $\angle AGF$ respectively, and $\angle CGD = \angle AGD$, we know that $\angle OGI = \angle NGL$. Also, $NL = OI$ and both $\angle OIG$ and $\angle GLN$ are right, so $\triangle OGI \cong \triangle NGL \cong \triangle OGH \cong \triangle NGK$. Now $\triangle OCH \cong \triangle OCJ$ by ASA congruence. Likewise $\triangle ODI \cong \triangle ODJ$, $\triangle NLF \cong \triangle NMF$, and $\triangle NAM \cong \triangle NAK$. All of these triangles have an altitude of common length, the inradius, which we will call r. The area of $\triangle CDG$ is the same as the area of $\triangle GFA$ as the medians dissect a triangle into six smaller triangles all of the same area. Thus:

$$(2)(\frac{1}{2}r)GH + (2)(\frac{1}{2}r)JD + (2)(\frac{1}{2}r)CJ = (2)(\frac{1}{2}r)GK + (2)(\frac{1}{2}r)AM + (2)(\frac{1}{2}r)MF$$

Since $GH = GK$:

$$\begin{aligned} (r)JD + (r)CJ &= (r)AM + r(MF) \\ JD + CJ &= AM + MF \\ AF &= CD \end{aligned}$$

which implies that $AB = AC = CB$ and thus the triangle is equilateral.

Solution 2 by: Shotaro Makisumi (9/CA)

Since the centroid divides each median into segments of proportion $1 : 2$, each of the six small triangles has a base that is half of and a height a third of $\triangle ABC$, and so they all have the same area. We know that the radii of the incircles of $\triangle AFG$, $\triangle BDG$, and $\triangle CDG$ are equal. Since $2A = rp$, where A is the area of the triangle, p is the perimeter, and r is the radius of the incircle, are all equal, these triangles all have equal perimeter. That is,

$$CD + CG + DG = BD + BG + DG = AF + AG + FG \qquad (1)$$

But $CD = BD$, so $CG = BG$. By SSS congruence, $\triangle CDG \cong \triangle BDG$. This implies $\angle CDG = 90°$.

We let $x = FG$ and $\theta = m\angle CGD = m\angle AGF$. Then we have $CG = 2x$. Since $\triangle CDG$ is a right triangle, $CD = 2x \sin\theta$ and $DG = 2x\cos\theta$, and so $AG = 2DG = 4x\cos\theta$. We apply the Law of Cosines on $\triangle AFG$:

$$\begin{aligned}(AF)^2 &= (FG)^2 + (AG)^2 - 2(FG)(AG)\cos(m\angle AGF)\\ (AF)^2 &= x^2 + (4x\cos\theta)^2 - 2x(4x\cos\theta)\cos\theta\\ (AF)^2 &= x^2 + 8x^2\cos^2\theta\\ AF &= x\sqrt{1+8\cos^2\theta}\end{aligned}$$

Now we can rewrite the second equality of (1) as follows:

$$2x + 2x\cos\theta + 2x\sin\theta = x\sqrt{1+8\cos^2\theta} + 4x\cos\theta + x$$

Since $x \neq 0$, we can divide through by x and simplify:

$$\begin{aligned}1 + 2\sin\theta - 2\cos\theta &= \sqrt{1+8\cos^2\theta}\\ 4\sin^2\theta + 4\sin\theta + 1 + 4\cos^2\theta - 4\cos\theta - 8\sin\theta\cos\theta &= 1 + 8\cos^2\theta\\ -2\cos^2\theta - \cos\theta + 1 + \sin\theta - 2\sin\theta\cos\theta &= 0\\ (\sin\theta + \cos\theta + 1)(1 - 2\cos\theta) &= 0\end{aligned}$$

This is satisfied when $\sin\theta + \cos\theta + 1 = 0$ or $1 - 2\cos\theta = 0$. For $\theta \in (0^\circ, 90^\circ)$, the former has no solution, since $\sin\theta > 0$ and $\cos\theta > 0$. We solve the second equation to obtain

$$\cos\theta = \frac{1}{2}$$

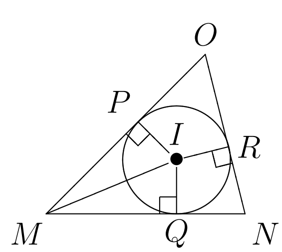
Finally, $AG = 4x\cos\theta = 2x = CG = BG$. Since the longer portions of the medians are congruent, the shorter portions are also congruent, and all six smaller triangles are congruent. This occurs only if $\triangle ABC$ is equilateral.

Q.E.D.

Solution 3 by: Dan Li (10/CA)

Lemma 5.1. *The inradius, r, of a triangle with sides m, n, o and angle μ opposite the side of length m is* $r = \left(\dfrac{n+o-m}{2}\right)\left(\tan\dfrac{\mu}{2}\right)$.

Proof. Let the triangle be $\triangle MNO$, with $NO = m$, $MN = n$, $MO = o$, and $\angle NMO = \mu$. Let the incenter of $\triangle MNO$ be I. Let the points of tangency on $\overline{MO}$, $\overline{MN}$, $\overline{NO}$ be P, Q, R, respectively.



Because I is equidistant from $\overline{MO}$ and $\overline{MN}$ ($IP = IQ = r$), it lies on the bisector of $\angle NMO$. Therefore,

$$\angle IMQ = \frac{\angle NMO}{2} = \frac{\mu}{2} \tag{2}$$

Because the segments from one point to two points of tangency have equal length (e.g. $MP = MQ$),

$$\begin{aligned} MQ &= n - QN = n - RN = n - (m - RO) \\ &= n - m + PO = n - m + (o - MP) = n - m + o - MQ \qquad (3) \\ 2MQ &= n + o - m \qquad (4) \\ MQ &= \frac{n+o-m}{2} \qquad (5) \end{aligned}$$

Thus,

$$r = IQ = MQ(\tan\angle IMQ) = \left(\frac{n+o-m}{2}\right)\left(\tan\frac{\mu}{2}\right) \tag{6}$$

□

Let $i = GD$, $j = GE$, $k = GF$, $a = FA$, $b = EA$, $c = DB$. Because the distance from the centroid (G) to a vertex is twice the distance from the centroid to the midpoint of the side opposite the vertex, $GA = 2i$, $GB = 2j$, $GC = 2k$. By the definition of median, $FB = a$, $EC = b$, $DC = c$.

Let $\alpha = \angle CGD = \angle AGF$. Because the inradii of $\triangle CGD$ and $\triangle AGF$ are equal and by Lemma 5.1,

$$\begin{aligned} \left(\frac{CG+GD-DC}{2}\right)\left(\tan\frac{\angle CGD}{2}\right) &= \left(\frac{AG+GF-FA}{2}\right)\left(\tan\frac{\angle AGF}{2}\right) \qquad (7) \\ \left(\frac{2k+i-c}{2}\right)\left(\tan\frac{\alpha}{2}\right) &= \left(\frac{2i+k-a}{2}\right)\left(\tan\frac{\alpha}{2}\right) \qquad (8) \\ 2k+i-c &= 2i+k-a \qquad (9) \\ k+a &= i+c \qquad (10) \end{aligned}$$

It is well-known that "all the medians together divide [a triangle] into six equal parts" (http://mathworld.wolfram.com/TriangleCentroid.html). Therefore, the areas of $\triangle AGF$, $\triangle CGD$, and $\triangle BGD$ are equal. It is similarly well-known that the product of the semi-perimeter and the inradius equals the area of a triangle (see (7) at `http://mathworld.wolfram.com/TriangleArea.html`; a proof is given at `http://mathworld.wolfram.com/Inradius.html`). Since the inradii of the three triangles are equal (since their incircles are congruent) and the areas are equal, their semiperimeters must be equal. Therefore,

$$\frac{2i+k+a}{2} = \frac{2k+i+c}{2} = \frac{2j+i+c}{2} \tag{11}$$

The second and third parts of (11) result in

$$\begin{aligned} \frac{2k+i+c}{2} &= \frac{2j+i+c}{2} && (12) \\ k &= j && (13) \end{aligned}$$

Therefore, $\triangle EGC \cong \triangle FGB$ by SAS ($EG = j = k = FG$, $CG = 2k = 2j = BG$, $\angle EGC \cong \angle FGB$). Hence, $EC = FB$ and

$$b = a \tag{14}$$

The first and second parts of (11) yield

$$\begin{aligned} \frac{2i+k+a}{2} &= \frac{2k+i+c}{2} && (15) \\ k+c &= i+a && (16) \end{aligned}$$

Subtracting (10) from (16) yields

$$\begin{aligned} c-a &= a-c && (17) \\ c &= a && (18) \end{aligned}$$

Combining (14) and (18),

$$\begin{aligned} a &= b = c \\ 2a &= 2b = 2c \\ AB &= AC = BC \end{aligned}$$

Hence, $\triangle ABC$ is equilateral.