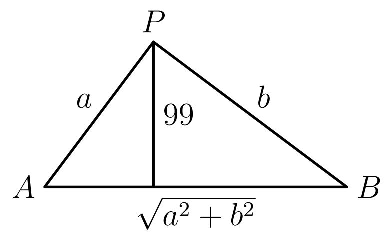

1/3/21. Let $ABCD$ be a convex quadrilateral with $AC \perp BD$, and let P be the intersection of AC and BD. Suppose that the distance from P to AB is 99, the distance from P to BC is 63, and the distance from P to CD is 77. What is the distance from P to AD?



Denote by a the distance from A to P. Likewise, let b, c, and d be the distances from P to B, C, and D. Since $AC \perp BD$, the triangles APB, BPC, CPD, and DPA all have right angles at P, and the distance from P to a side is the length of the altitude of the respective triangle. Consider triangle PAB at right. Then 99 (the distance from P to AB) is the length of the altitude, and computing the area of PAB in two ways gives

$$2(\text{Area}) = ab = 99\sqrt{a^2+b^2}.$$

Squaring and rearranging gives

$$99^2 = \frac{a^2b^2}{a^2+b^2}.$$

The reciprocal of this equation will be more useful to us:

$$\frac{1}{99^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

Denoting the distance from P to AD (the value we want to find) by x, we likewise have

$$\frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{63^2}$$
$$\frac{1}{c^2} + \frac{1}{d^2} = \frac{1}{77^2}$$
$$\frac{1}{d^2} + \frac{1}{a^2} = \frac{1}{x^2}.$$

Combining these in pairs we get

$$\frac{1}{99^2} + \frac{1}{77^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = \frac{1}{63^2} + \frac{1}{x^2},$$

or

$$\frac{1}{x^2} = \frac{1}{99^2} + \frac{1}{77^2} - \frac{1}{63^2} = \frac{1}{231^2}.$$

Therefore $x = \boxed{231}$.

2/3/21. Find, with proof, a positive integer n such that

$$\frac{(n+1)(n+2)\cdots(n+500)}{500!}$$

is an integer with no prime factors less than 500.

We can rewrite our expression as

$$\begin{aligned}\frac{(n+1)(n+2)\cdots(n+500)}{500!} &= \left(\frac{n+1}{1}\right)\left(\frac{n+2}{2}\right)\left(\frac{n+3}{3}\right)\cdots\left(\frac{n+500}{500}\right)\\ &= \left(\frac{n}{1}+1\right)\left(\frac{n}{2}+1\right)\left(\frac{n}{3}+1\right)\cdots\left(\frac{n}{500}+1\right)\\ &= \prod_{j=1}^{500}\left(\frac{n}{j}+1\right).\end{aligned}$$

We claim that $n = \boxed{(500!)^2}$ satisfies the condition of the problem. To prove this, notice that for every j with $1 \le j \le 500$, the quantity $\dfrac{(500!)^2}{j}$ is an integer and is furthermore divisible by 500!. Therefore $\frac{(500!)^2}{j}$ is divisible by every prime less than 500. So for any prime $p < 500$,

$$\prod_{j=1}^{500}\left(\frac{(500!)^2}{j}+1\right) \equiv \prod_{j=1}^{500} 1 \equiv 1 \pmod{p},$$

proving the result.

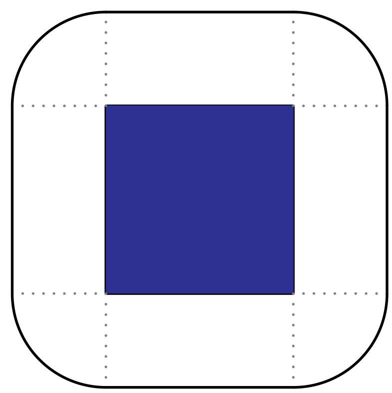
3/3/21. We are given a rectangular piece of white paper with length 25 and width 20. On the paper we color blue the interiors of 120 disjoint squares of side length 1 (the sides of the squares do not necessarily have to be parallel to the sides of the paper). Prove that we can draw a circle of diameter 1 on the remaining paper such that the entire interior of the circle is white.

If we are able to place our circle entirely on the paper such that it does not overlap any squares, the center of the circle must have two properties:

I. The center of the circle must be a distance at least 1/2 from the boundary of the paper.

II. The center of the circle must be outside the neighborhood of distance 1/2 of every blue square.

The area of the region that satisfies condition (I) is $24 \cdot 19$.



Consider the diagram to the right. The neighborhood of distance 1/2 of a square is a region consisting of the square (of area 1), 4 rectangles each with dimensions $1 \times \frac{1}{2}$, and four quarter-circles each of radius $\frac{1}{2}$. Thus, the total area of each neighborhood is

$$3 + \frac{\pi}{4}.$$

The circle will not intersect the square if and only if the center of the circle is outside this region.

Therefore, the total restricted region under condition (II) is the union of the 120 neighborhoods of the blue squares. This area is at most $120\left(3 + \frac{\pi}{4}\right) = 360 + 30\pi$ (it may be less if the neighborhoods overlap). Subtracting this area from the allowed area under condition (I) gives an overall allowed area of at least

$$(24)(19) - (360 + 30\pi) = 96 - 30\pi = 6(16 - 5\pi).$$

Since $16 > 5\pi$, the area of the allowed region is positive, and a circle of diameter 1 centered at any point in this region will lie entirely on the page and will not overlap any of the squares.

4/3/21. Let a and b be positive integers such that all but 2009 positive integers are expressible in the form $ma+nb$, where m and n are nonnegative integers. If 1776 is one of the numbers that is not expressible, find $a+b$.

We begin by considering the question: Given general a and b, what is the set of numbers which are not expressible as $ma+nb$ where m and n are nonnegative integers? First notice that if $g = \gcd(a, b) > 1$, then g is a divisor of any $ma+nb$, so this set is infinite. We may assume, then, that a and b are relatively prime. Further, since we are trying to find the sum $a+b$, we may assume $a < b$.

The following is a known theorem, related to the Frobenius Coin Problem and Chicken McNugget Problem.

Theorem 1: Let a and b be relatively prime positive integers, $N = ab - a - b$, and let S be the set of nonnegative integers which are not expressible in the form $an+bm$ for nonnegative m, n. Then

1. Every element of S is less than or equal to N, and
2. If $0 \le x, y \le N$ and $x+y = N$, then exactly one of x or y is in S.

(For completeness, we include a proof of this theorem at the end of the solution.)

From this theorem we see that S contains exactly half of the integers from 0 to N, so has size $\frac{N+1}{2} = \frac{(a-1)(b-1)}{2}$ (notice that N is odd). By assumption, there are 2009 elements of S, so $N = 2 \cdot 2009 - 1 = 4017$ and

$$(a-1)(b-1) = 4018 = 2 \cdot 7^2 \cdot 41.$$

The possible values of $a < b$ are

a	2	3	8	15	42	50
b	4019	2010	575	288	99	83

Of these, only $(2, 4019)$, $(8, 575)$, and $(50, 83)$ are relatively prime.

We know that, for any of these pairs, exactly one of 1776 and $N - 1776 = 4017 - 1776 = 2241$ is expressible as a sum of the form $ma+nb$. If we let $(a, b) = (50, 83)$, then we notice

$$2241 = 0 \cdot 50 + 27 \cdot 83,$$

proving that $(a, b) = (50, 83)$ satisfies the criteria. Checking the other two pairs, we see

$$1776 = 888 \cdot 2 + 0 \cdot 4019$$
$$1776 = 222 \cdot 8 + 0 \cdot 575,$$

showing that these are not solutions. Therefore

$$a + b = 50 + 83 = \boxed{133}.$$

Proof of Theorem 1: Let

$$T = \{ma + nb \mid 0 \le m < b \text{ and } 0 \le n < a\}.$$

By the Chinese Remainder Theorem, this set spans the residue classes modulo ab: each residue class contains exactly one element of T. Every element in T is less than $2ab$. Notice that if $c \in S$, then c is not in T (although the converse is false).

For any integer $c > ab$, either $c \in T$ or $c = c' + kab$ for $c' \in T$ and some $k \ge 1$, so $c \notin S$. Therefore S is a subset of the positive integers less than ab, so is finite.

Notice that the largest element of T is $a(b-1) + b(a-1) = 2ab - a - b$. If $2ab - a - b < c < 2ab$, then the element of T in the residue class modulo ab of c is $c - ab$. Therefore T contains all integers d with $ab - a - b < d < ab$, so the largest element of S is at most $ab - a - b$. This proves the first part of the theorem.

Next, assume that $0 < c < N$ can be expressed as $c = am + bn$ for nonnegative m and n (so $c \in T$). Clearly $m < b$ and $n < a$. Then letting $c' = N - c$, we have

$$c' = N - c = (ab - a - b) - (am + bn) = a(b - m - 1) + b(-n - 1).$$

This tells us that

$$c' + ab = a(b - m - 1) + b(a - n - 1) \in T,$$

so the residue class of c' mod (ab) is represented in T by $c' + ab$. This tells us $c' \notin T$.

Furthermore, we can reverse this argument: If $c' \notin T$ and $0 \le c' \le N$, then $c' + ab \in T$, so

$$c' = am + bn - ab,$$

for nonnegative $m < b$ and $n < a$. Therefore

$$c = ab-a-b-c' = ab-a-b-(am+bn-ab) = 2ab-a-b-am-bn = a(b-m-1)+b(a-n-1).$$

This is an expression of $c = N - c'$ as a nonnegative combination of a and b, proving

$$c \in T \iff N - c \notin T,$$

for all $0 \le c \le N$. From this we can conclude that if $0 \le c \le N$, then

$$c \in S \iff N - c \notin S.$$

5/3/21. The sequences (a_n), (b_n), and (c_n) are defined by $a_0 = 1$, $b_0 = 0$, $c_0 = 0$, and

$$a_n = a_{n-1} + \frac{c_{n-1}}{n}, \quad b_n = b_{n-1} + \frac{a_{n-1}}{n}, \quad c_n = c_{n-1} + \frac{b_{n-1}}{n}$$

for all $n \geq 1$. Prove that

$$\left| a_n - \frac{n+1}{3} \right| < \frac{2}{\sqrt{3n}}$$

for all $n \geq 1$.

Solution 1. Computing the first few values of a_n, b_n, and c_n, we find $a_1 = 1$, $b_1 = 1$, $c_1 = 0$, $a_2 = 1$, $b_2 = 1 + \frac{1}{2}$, $c_2 = \frac{1}{(1\cdot 2)}$, and

$$a_3 = 1 + \frac{1}{1\cdot 2\cdot 3}, \quad b_3 = 1 + \frac{1}{2} + \frac{1}{3}, \quad c_3 = \frac{1}{1\cdot 2} + \frac{1}{1\cdot 3} + \frac{1}{2\cdot 3}.$$

By a straightforward induction argument, we can prove the following: For a set X of real numbers, let $\pi(X)$ denote the product of the elements of X. Set $\pi(\emptyset) = 1$. Then

$$a_n = \sum_{\substack{S\subseteq\{1,2,\dots,n\} \\ |S|\equiv 0 \pmod 3}} \frac{1}{\pi(S)}, \qquad b_n = \sum_{\substack{S\subseteq\{1,2,\dots,n\} \\ |S|\equiv 1 \pmod 3}} \frac{1}{\pi(S)}, \qquad c_n = \sum_{\substack{S\subseteq\{1,2,\dots,n\} \\ |S|\equiv 2 \pmod 3}} \frac{1}{\pi(S)}.$$

The inequality in the problem is easily checked for $n = 1$ and $n = 2$, so henceforth, assume that $n \geq 3$. For $0 \leq k \leq n$, let

$$s_k = \sum_{\substack{S\subseteq\{1,2,\dots,n\} \\ |S|=k}} \frac{1}{\pi(S)}.$$

Then $a_n = s_0 + s_3 + s_6 + \cdots$, and the generating function for the s_k is

$$f(x) = s_0 + s_1 x + s_2 x^2 + \cdots + s_n x^n = \left(1 + \frac{x}{1}\right)\left(1 + \frac{x}{2}\right)\cdots\left(1 + \frac{x}{n}\right).$$

Let $\omega = e^{2\pi i/3}$. Then setting $x = 1$, ω, ω^2, respectively, we obtain the equations

$$\begin{aligned} s_0 + s_1 + s_2 + \cdots + s_n &= f(1), \\ s_0 + s_1\omega + s_2\omega^2 + \cdots + s_n\omega^n &= f(\omega), \\ s_0 + s_1\omega^2 + s_2\omega^4 + \cdots + s_n\omega^{2n} &= f(\omega^2). \end{aligned}$$

Adding, we get

$$3a_n = f(1) + f(\omega) + f(\omega^2),$$

so

$$a_n = \frac{f(1) + f(\omega) + f(\omega^2)}{3}.$$

Now,

$$f(1) = \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\cdots\left(1 + \frac{1}{n}\right) = \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n} = n + 1.$$

To get an estimate for $f(\omega)$, for $1 \le k \le n$, let

$$1 + \frac{\omega}{k} = r_k e^{i\theta_k},$$

where $r_k > 0$ and $0 < \theta_k < \pi$. Then

$$\begin{aligned} r_k^2 &= \left|1 + \frac{\omega}{k}\right|^2 \\ &= \left|1 + \frac{-1 + \sqrt{3}i}{2k}\right|^2 \\ &= \left(1 - \frac{1}{2k}\right)^2 + \frac{3}{4k^2} \\ &= 1 - \frac{1}{k} + \frac{1}{4k^2} + \frac{3}{4k^2} \\ &= 1 - \frac{1}{k} + \frac{1}{k^2} \\ &= \frac{k^2 - k + 1}{k^2}, \end{aligned}$$

so

$$\begin{aligned} |f(\omega)|^2 &= \left|\left(1 + \frac{\omega}{1}\right)\left(1 + \frac{\omega}{2}\right)\cdots\left(1 + \frac{\omega}{n}\right)\right|^2 \\ &= r_1^2 r_2^2 \cdots r_n^2 \\ &= \prod_{k=1}^{n} \frac{k^2 - k + 1}{k^2} \\ &= \frac{1}{1^2}\cdot\frac{3}{2^2}\cdot\frac{7}{3^2}\cdots\frac{n^2 - n + 1}{n^2} \\ &= \frac{3}{1^2}\cdot\frac{7}{2^2}\cdot\frac{13}{3^2}\cdots\frac{n^2 - n + 1}{(n-1)^2}\cdot\frac{1}{n^2} \\ &= \frac{1}{n^2}\prod_{k=1}^{n-1}\frac{k^2 + k + 1}{k^2}. \end{aligned}$$

Note that

$$\frac{k^2+k+1}{k^2} < \frac{k}{k-1}$$

for all positive integers $k \geq 2$, since this is equivalent to $k^3 - 1 < k^3$. Hence,

$$\begin{aligned} |f(\omega)|^2 &= \frac{1}{n^2} \prod_{k=1}^{n-1} \frac{k^2+k+1}{k^2} \\ &= \frac{3}{n^2} \prod_{k=2}^{n-1} \frac{k^2+k+1}{k^2} \\ &< \frac{3}{n^2} \prod_{k=2}^{n-1} \frac{k}{k-1} \\ &= \frac{3(n-1)}{n^2} \\ &< \frac{3n}{n^2} = \frac{3}{n}, \end{aligned}$$

so

$$|f(\omega)| < \sqrt{\frac{3}{n}}.$$

Since $f(x)$ is a polynomial in x with real coefficients,

$$f(\omega^2) = f(\overline{\omega}) = \overline{f(\omega)},$$

so

$$|f(\omega^2)| = |\overline{f(\omega)}| = |f(\omega)|.$$

Hence,

$$\left| a_n - \frac{n+1}{3} \right| = \left| \frac{f(\omega) + f(\omega^2)}{3} \right| \leq \frac{|f(\omega)| + |f(\omega^2)|}{3} < \frac{2}{3}\sqrt{\frac{3}{n}} = \frac{2}{\sqrt{3n}}.$$

Solution 2. By a straightforward induction argument, $a_n + b_n + c_n = n + 1$ for all $n \geq 0$. Let

$$d_n = (a_n - b_n)^2 + (a_n - c_n)^2 + (b_n - c_n)^2$$

for all $n \geq 0$. Then for all $n \geq 1$,

$$\begin{aligned}
d_n &= (a_n - b_n)^2 + (a_n - c_n)^2 + (b_n - c_n)^2 \\
&= \left(a_{n-1} + \frac{c_{n-1}}{n} - b_{n-1} - \frac{a_{n-1}}{n}\right)^2 \\
&\quad + \left(a_{n-1} + \frac{c_{n-1}}{n} - c_{n-1} - \frac{b_{n-1}}{n}\right)^2 \\
&\quad + \left(b_{n-1} + \frac{a_{n-1}}{n} - c_{n-1} - \frac{b_{n-1}}{n}\right)^2 \\
&= \frac{2(n^2 - n + 1)}{n^2}(a_{n-1}^2 + b_{n-1}^2 + c_{n-1}^2 - a_{n-1}b_{n-1} - a_{n-1}c_{n-1} - b_{n-1}c_{n-1}) \\
&= \frac{n^2 - n + 1}{n^2}[(a_{n-1} - b_{n-1})^2 + (a_{n-1} - b_{n-1})^2 + (b_{n-1} - c_{n-1})^2] \\
&= \frac{n^2 - n + 1}{n^2} d_{n-1}.
\end{aligned}$$

Hence,

$$d_n = \left(\prod_{k=1}^{n} \frac{k^2 - k + 1}{k^2}\right) d_0.$$

Since $d_0 = 2$, by the same argument as in Solution 1, $d_n < 6/n$ for all $n \geq 1$.

Now we introduce a lemma.

Lemma. Let x, y, z, and ϵ be real numbers such that $x + y + z = 0$ and

$$(x - y)^2 + (x - z)^2 + (y - z)^2 < \epsilon.$$

Then

$$\max\{|x|, |y|, |z|\} < \frac{\sqrt{2\epsilon}}{3}.$$

Proof. Expanding, we get

$$\begin{aligned}
(x - y)^2 + (x - z)^2 + (y - z)^2 &= 2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz \\
&= 2x^2 + 2(y + z)^2 - 4yz - 2x(y + z) - 2yz \\
&= 6x^2 - 6yz.
\end{aligned}$$

But

$$yz = \frac{(y + z)^2 - (y - z)^2}{4} = \frac{x^2 - (y - z)^2}{4},$$

so

$$(x-y)^2 + (x-z)^2 + (y-z)^2 = 6x^2 - \frac{3[x^2 - (y-z)^2]}{2}$$
$$= \frac{9x^2 + 3(y-z)^2}{2}.$$

Then $9x^2 \le 9x^2 + 3(y-z)^2 < 2\epsilon$, so $|x| < \frac{\sqrt{2\epsilon}}{3}$. Similarly, $|y| < \frac{\sqrt{2\epsilon}}{3}$ and $|z| < \frac{\sqrt{2\epsilon}}{3}$. ■

Take $x = a_n - \frac{n+1}{3}$, $y = b_n - \frac{n+1}{3}$, and $z = c_n - \frac{n+1}{3}$. Then $x+y+z = a_n+b_n+c_n-(n+1) = 0$, and

$$(x-y)^2 + (x-z)^2 + (y-z)^2 = (a_n - b_n)^2 + (a_n - c_n)^2 + (b_n - c_n)^2 = d_n < \frac{6}{n},$$

so by the lemma,

$$\left| a_n - \frac{n+1}{3} \right| = |x| < \frac{\sqrt{2 \cdot 6/n}}{3} = \frac{2}{\sqrt{3n}}.$$

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Problem 4/3/21 is loosely based on Problem A5 from the 1971 Putnam Competition.

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