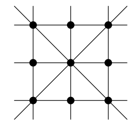
USA Mathematical Talent Search

Round 1 Solutions

Year 22 — Academic Year 2010–2011

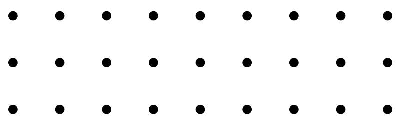
www.usamts.org

1/1/22.



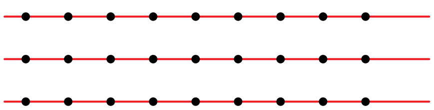
Given a set S of points in the plane, a line is called *happy* if it contains at least 3 points in S. For example, if S is the 3×3 grid of points shown at right, then there are 8 happy lines as shown.

(a) If S is the 3×9 grid shown below, how many happy lines are there?

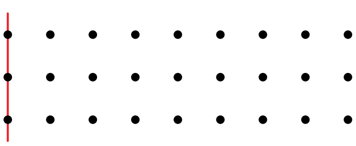


(b) Find, with proof, a set S (in the plane) with 27 points that has exactly 49 happy lines.

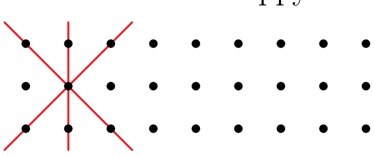
(a) We count the happy lines in sets. First notice that there are three horizontal happy lines.



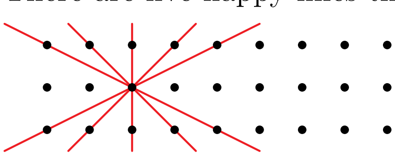
Any other happy line must contain a point in the second row. We count the remaining happy lines by counting how many pass through a given middle point. There is exactly one happy line through the leftmost point.



There are three happy lines through the second point.



There are five happy lines through the third point.



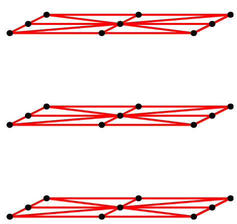
The other points have 7, 9, 7, 5, 3, and 1 happy lines respectively.

The total number of happy lines in the diagram is

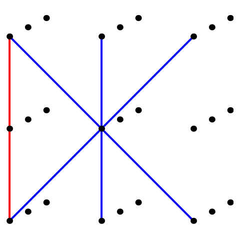
$$3 + (1 + 3 + 5 + 7 + 9 + 7 + 5 + 3 + 1) = \boxed{44}.$$

(b) There are many solutions to this problem. For instance, we can take a $3 \times 3 \times 3$ cube of dots in space and project its $3^3 = 27$ vertices onto the plane in such a way that all of the lines in space which contain three vertices are taken to distinct lines in the plane and no new lines are created [1]. We let S be the projection and count the happy lines.

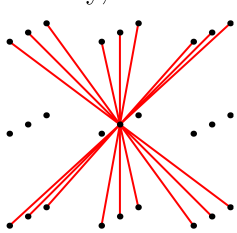
First we may count the lines which all lie on "horizontal" planes. This gives 24 lines.



The remaining lines must intersect each of the three horizontal planes one time, so must contain one vertex on each plane. We enumerate these by considering which of the 9 vertices are on the line. By the "center square" we will mean the nine vertices on the central horizontal plane. Each of the four corner vertices of the central square meet only one non-horizontal line (in red, below). Each of the four edge vertices of the center square are found on three non-horizontal lines (in blue below).



Finally, there are nine non-horizontal lines through the center vertex.



This gives a total of $24 + 4 \cdot 1 + 4 \cdot 3 + 9 = 49$ happy lines.

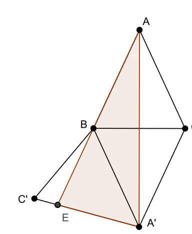
[1] We can always do this: the noncollinear subsets of 3 points from our 27 points generate a set of at most $\binom{27}{3}$ planes. If we project to a plane that is not perpendicular to any plane in this set, we will not create any new lines. Further, if we project to a plane that is not perpendicular to any of our existing lines, we will not lose any lines upon projection.

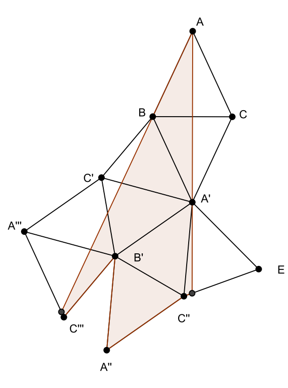
2/1/22. You're at vertex A of triangle ABC, where $\angle B = \angle C = 65^\circ$. The sides of the triangle are perfectly reflective; if you shoot a laser from A to the midpoint of $\overline{BC}$, it will reflect once and return to A. Suppose you fire at a point on $\overline{BC}$ other than its midpoint, and the beam still returns to A after reflecting some number of times. What is the smallest number of reflections the beam can make before returning to A? What is the smallest angle between $\overline{AB}$ and the initial beam that produces this number of reflections?

We label the edge opposite A as a, the edge opposite B as b and the edge opposite C as c. We will measure the initial angle θ from $\overline{AB}$, where $0 < \theta < 50^\circ$.

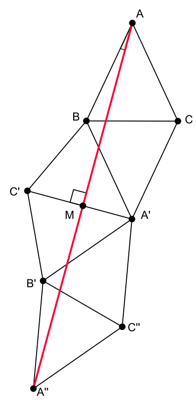
Since the room is perfectly reflective, standing at point A we see an infinite region in which many copies of A, B, and C appear. We are looking for the minimum number of walls that the beam may cross before returning to A. Furthermore, from the point of view of the beam, it is traveling in a straight line and passing through a number of consecutive isosceles chambers.

Notice that the solution set is symmetric, so any closed path with angle $\theta > 25^\circ$ is the horizontal reflection of a closed path with angle $50 - \theta$. Therefore we may assume that the path which passes the fewest faces meets a and then c. The sum of the three acute angles at B is $3 \cdot 65^\circ = 195^\circ > 180^\circ$, so the beam must subsequently pass through edge b. The shaded region shows where the beam may pass.





The continuation of the beam's possible paths at left includes the set of all chambers that the beam can visit in five or fewer reflections with an initial angle less that 25°. Since there is one reflection of A other than A', the minimum number of reflections must be five. Since B and C' are both reflex angles, the beam cannot reach A''.



We now know that the minimum number of reflections is 5 and is achieved by the path at left. Let M denote the midpoint of the beam $\overline{AA''}$. By symmetry M lies on $\overline{CA'}$, and the beam intersects this wall with angle 90°. Now consider the quadrilateral $AMC'B$. This quadrilateral has total angle

$$360^\circ = A + 90^\circ + 65^\circ + (3 \cdot 65^\circ) = A + 350^\circ.$$

Therefore we must fire the beam at an angle 10° in order to return in the fewest number of reflections (other than firing at the midpoint of $\overline{BC}$).

3/1/22. Find $c > 0$ such that if r, s, and t are the roots of the cubic

$$f(x) = x^3 - 4x^2 + 6x + c,$$

then

$$1 = \frac{1}{r^2+s^2} + \frac{1}{s^2+t^2} + \frac{1}{t^2+r^2}.$$

We know that $x^3 - 4x^2 + 6x + c = (x-r)(x-s)(x-t)$, therefore

$$r+s+t = 4, \tag{1}$$
$$rs+rt+st = 6, \tag{2}$$
$$rst = -c. \tag{3}$$

Next we compute the sum of the squares of the roots using (1) and (2) above:

$$r^2+s^2+t^2 = (r+s+t)^2 - 2(rs+rt+st) = 4^2 - 2(6) = 4. \tag{4}$$

Thus, using (4), our identity becomes

$$\begin{aligned}
1 &= \frac{1}{r^2+s^2} + \frac{1}{s^2+t^2} + \frac{1}{t^2+r^2} \\
&= \frac{1}{4-t^2} + \frac{1}{4-r^2} + \frac{1}{4-s^2} \\
&= \frac{(4-r^2)(4-s^2)+(4-s^2)(4-t^2)+(4-t^2)(4-r^2)}{(4-r^2)(4-s^2)(4-t^2)} \\
&= \frac{48 - 8(r^2+s^2+t^2) + r^2s^2 + r^2t^2 + s^2t^2}{64 - 16(r^2+s^2+t^2) + 4(r^2s^2+r^2t^2+s^2t^2) - r^2s^2t^2}.
\end{aligned} \tag{5}$$

Substituting using (3) and (4) gives

$$1 = \frac{48 - 8(4) + r^2s^2 + r^2t^2 + s^2t^2}{64 - 16(4) + 4(r^2s^2+r^2t^2+s^2t^2) - c^2} = \frac{16 + r^2s^2 + r^2t^2 + s^2t^2}{4(r^2s^2+r^2t^2+s^2t^2) - c^2}. \tag{6}$$

Next, observe that by (2)

$$36 = (rs+rt+st)^2 = r^2s^2 + r^2t^2 + s^2t^2 + 2rst(r+s+t) = r^2s^2 + r^2t^2 + s^2t^2 - 8c,$$

so that

$$r^2s^2 + r^2t^2 + s^2t^2 = 36 + 8c. \tag{7}$$

Combining (6) and (7) gives

$$1 = \frac{16 + (36+8c)}{4(36+8c) - c^2} = \frac{52+8c}{144+32c-c^2},$$

and hence $c^2 - 24c - 92 = 0$. The solutions to this quadratic are

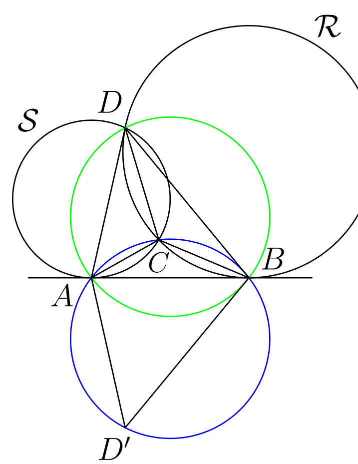
$$c = \frac{24 \pm \sqrt{24^2 + 4(92)}}{2}$$

and the only positive solution is $\boxed{c = 12 + 2\sqrt{59}}$.

4/1/22. Sasha has a compass with fixed radius s and Rebecca has a compass with fixed radius r. Sasha draws a circle (with his compass) and Rebecca then draws a circle (with her compass) that intersects Sasha's circle twice. We call these intersection points C and D.

Charlie draws a common tangent to both circles, meeting Sasha's circle at point A and Rebecca's circle at point B, and then draws the circle passing through A, B, and C. Prove that the radius of Charlie's circle does not depend on where Sasha and Rebecca choose to draw their circles, or which of the two common tangents Charlie draws.

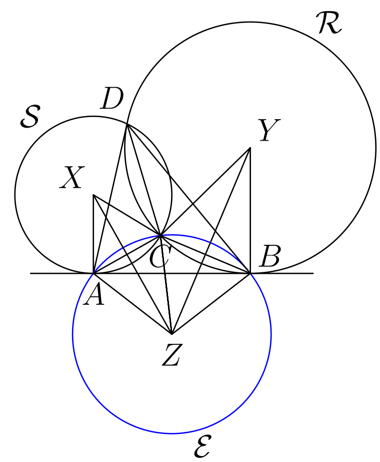
Let Sasha's circle be $\mathcal{S}$ and Rebecca's be $\mathcal{R}$. Once $\mathcal{S}$ and $\mathcal{R}$ are chosen, Charlie has two choices for the common tangent. Suppose he chooses the tangent such that the tangent is closer to C than to D, as shown in the diagram at the right. Had he chosen the other tangent, the resulting circumcircle of $\triangle ABC$ would be congruent to the shown circumcircle of $\triangle ABD$, by symmetry. So, we will show that Charlie's two options for the common tangent produce the same circumradius by showing that the circumradii of $\triangle ABC$ and $\triangle ABD$ are equal. We do so by showing that the circumcircle of $\triangle ABC$ is the reflection of the circumcircle of $\triangle ABD$ over $\overleftrightarrow{AB}$.



Clearly the reflections of A and B over $\overleftrightarrow{AB}$ are on the circumcircle of $\triangle ABD$. We need only show that the reflection of D over $\overleftrightarrow{AB}$, which we call D', is on the circumcircle of $\triangle ABC$. Since angles $\angle ADC$ and $\angle CAB$ are both inscribed in arc $\widehat{AC}$ of $\mathcal{S}$, we have $\angle ADC = \angle CAB$. Similarly, we have $\angle CDB = \angle CBA$ because these two angles are inscribed in arc $\widehat{CB}$ of $\mathcal{R}$. Therefore, we have

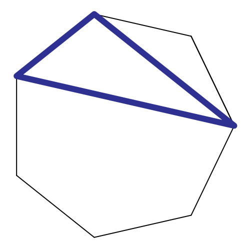
$$\angle ADB + \angle ACB = \angle ADC + \angle CDB + \angle ACB = \angle CAB + \angle CBA + \angle ACB = 180^\circ,$$

so $\angle ADB$ is supplementary to $\angle ACB$, which means $\angle AD'B$ and $\angle ACB$ are supplementary. Since D and C are on the same side of $\overleftrightarrow{AB}$, points D' and C are on opposite sides of $\overleftrightarrow{AB}$. Combining this with the fact that $\angle AD'B + \angle ACB = 180^\circ$, we know that $AD'BC$ is a cyclic quadrilateral, so D' is on the circumcircle of $\triangle ACB$.

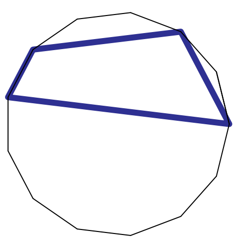


Let Charlie's circle be $\mathcal{E}$. Let X, Y, Z be the centers of $\mathcal{S}$, $\mathcal{R}$, $\mathcal{E}$, respectively. We will find the radius of $\mathcal{E}$ in terms of r and s. Since $\angle CAB$ is inscribed in $\widehat{AC}$ of $\mathcal{S}$, we have $\angle CXA = \widehat{AC} = 2\angle CAB$. Similarly, since $\angle CAB$ is inscribed in $\widehat{BC}$ of $\mathcal{E}$, we have $\angle CZB = \widehat{BC} = 2\angle CAB$. Therefore, we have $\angle CZB = \angle CXA$, so $\angle CZY = \angle CXZ$ (because $\overline{ZY}$ bisects $\angle CZB$ and $\overline{XZ}$ bisects $\angle CXA$). Similarly, we have $\angle BYC = 2\angle CBA = \angle CZA$, so $\angle CYZ = \angle CZX$. Combining $\angle CZY = \angle CXZ$ and $\angle CYZ = \angle CZX$, we have $\triangle CXZ \sim \triangle CZY$. Therefore, we have $\frac{CZ}{CX} = \frac{CY}{CZ}$, so $CZ^2 = CX \cdot CY = rs$, which means that the radius of $\mathcal{E}$ is $\sqrt{rs}$. We conclude that Charlie's circle has radius $\sqrt{rs}$ no matter where Sasha and Rebecca locate their circles (as long as the circles intersect).

5/1/22. A convex polygon P is called *peculiar* if: (a) for some $n \geq 3$, the vertices of P are a subset of the vertices of a regular n-gon with sides of length 1; (b) the center O of the n-gon lies outside of P; and (c) for every integer k with $0 < k \leq \frac{n}{2}$, the quantity $\frac{2k\pi}{n}$ is the measure of exactly one $\angle AOB$, where A and B are vertices of P. Find the number of non-congruent peculiar polygons.



We notice first that if n is even, then letting $k = \frac{n}{2}$ tells us we need one central angle of $\frac{2\pi n}{k} = \pi$. However if this is the case then the diameter between the two corresponding points is a subset of the peculiar polygon, so the center of the n-gon lies in P, which violates condition (b). Therefore n must be odd. First we give two examples of peculiar polygons. To the right we have a peculiar triangle for $n = 7$, and below we have a peculiar quadrilateral for $n = 13$.



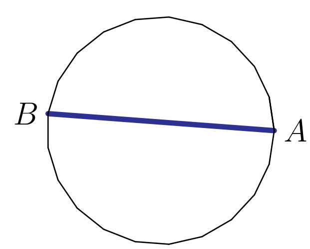
Let v be the number of vertices of P, and let $r = \frac{n-1}{2}$ (note r is an integer because n is odd). Each pair of vertices describes a unique central angle less than π. Since this set of angles is

$$\left\{\frac{2\cdot 1\pi}{n}, \frac{2\cdot 2\pi}{n}, \ldots, \frac{2\cdot r\pi}{n}\right\},$$

there are r such angles. Therefore $\binom{v}{2} = r$, or

$$n = v^2 - v + 1.$$

We intend to show that there are no peculiar polygons beyond the two given above.



Assume we have a peculiar polygon. This polygon must contain a pair of vertices, A and B, of maximal central angle $\angle AOB = \frac{(n-1)\pi}{n}$. The chord $\overline{AB}$ separates the remaining vertices of the n-gon into two sets of size $r-1$ and r. If any points from the larger set are also vertices of P, then the center will be interior to P. Therefore, all vertices of P must lie on the short side of $\overline{AB}$. In particular, $\overline{AB}$ is an edge of P.

We now relabel the vertices from A to B as $A = A_0, A_1, \ldots, A_r = B$. Next we look for a pair of vertices forming an angle of $\frac{(n-3)\pi}{n}$. Such a pair is only possible if one member of the pair is either A_0 or A_r. We may assume this is A_0 by congruence. Then, the other vertex must be A_{r-1}. Therefore P must have vertices A_0, A_{r-1}, and A_r.

In the case $v = 3$, these three vertices above describe the peculiar triangle. We now assume that $v > 3$.

Notice that, by the uniqueness condition of property (c), the points A_1 and A_{r-2} cannot be vertices of P since $\widehat{A_0A_1}$ and $\widehat{A_{r-2}A_{r-1}}$ are both congruent to $\widehat{A_{r-1}A_r}$.

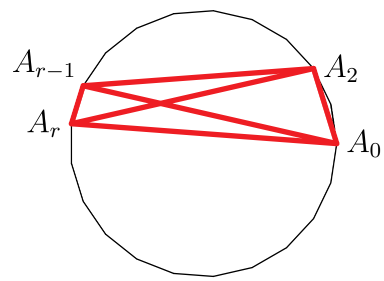
The only arcs of angle $\frac{(n-5)\pi}{n}$ that lie above $\overline{AB}$ are

$$\widehat{A_0A_{r-2}}, \quad \widehat{A_1A_{r-1}}, \quad \text{and} \quad \widehat{A_2A_r}.$$

Since A_1 and A_{r-2} are not vertices of P, we have that A_2 must be a vertex. Therefore, P must have the vertices A_0, A_2, A_{r-1}, and A_r. If $v = 4$ then $n = 4^2 - 4 + 1 = 13$ gives the peculiar quadrilateral above.

Now we assume that $v > 4$. We have the four vertices from above, and these four vertices span the six angles

$$\frac{2\pi}{n}, \frac{4\pi}{n}, \frac{(n-7)\pi}{n}, \frac{(n-5)\pi}{n}, \frac{(n-3)\pi}{n}, \text{ and } \frac{(n-1)\pi}{n}.$$



We claim that there is no way to add a vertex (or more) in order to get the angle $\frac{(n-9)\pi}{n}$ without creating another angle we have already constructed.

The only arcs of angle $\frac{(n-9)\pi}{n}$ which lie above $\overline{A_0A_r}$ are

$$\widehat{A_0A_{r-4}}, \quad \widehat{A_1A_{r-3}}, \quad \widehat{A_2A_{r-2}}, \quad \widehat{A_3A_{r-1}}, \text{ and } \widehat{A_4A_r}.$$

The only new vertex on this list not at an angle $\frac{2\pi}{n}$ or $\frac{4\pi}{n}$ from an existing vertex is A_{r-4}. Therefore any peculiar polygon must contain this point as well.

If $v = 5$ then $r = 10$, so the vertices are A_0, A_2, A_6, A_9, and A_{10}. However this is not a possible peculiar polygon as the angle from A_2 to A_6 is the same as that from A_6 to A_{10}. There is no peculiar pentagon.

Assume that $v > 5$ and that we have the vertices A_0, A_2, A_{r-4}, A_{r-1}, and A_r. We need to place a point to give an angle of $\frac{(n-9)\pi}{n}$. The only possiblilites are

$$\widehat{A_0A_{r-5}}, \widehat{A_1A_{r-4}}, \widehat{A_2A_{r-3}}, \widehat{A_3A_{r-2}}, \widehat{A_4A_{r-1}}, \widehat{A_5A_r}.$$

None of these pairs is a subset of the current vertices since

$$r - 5 = \frac{v^2 - v}{2} - 5 > \frac{5^2 - 5}{2} - 5 = 5.$$

Furthermore, each of the new potential vertices is an angle of $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, or $\frac{6\pi}{n}$ from an existing vertex, so is not permissible. Therefore there are no peculiar polygons with 5 or more vertices.

By our construction, there is exactly one peculiar triangle, one peculiar quadrilateral, and zero other peculiar polygons, up to congruence. Therefore, the answer is $\boxed{2}$.

6/1/22. There are 50 people (numbered 1 to 50) and 50 identically wrapped presents around a table at a party. Each present contains an integer dollar amount from \$1 to \$50, and no two presents contain the same amount. Each person is randomly given one of the presents. Beginning with player #1, each player in turn does **one** of the following:

1. Opens his present and shows everyone the contents; **or**
2. If another player at the table has an open present, the player whose turn it is may swap presents with that player, and leave the table with the open present. The other player then immediately opens his new present and shows everyone the contents.

For example, the game could begin as follows:

- Player #1 opens his present. (The game must *always* begin this way, as there are no open presents with which to swap.)
- Player #2 decides to swap her present with Player #1. Player #2 takes the money from her newly acquired present and leaves the table. Player #1 opens his new present (which used to belong to Player #2).
- Player #3 opens her present. (Now Players #1 and #3 have open presents, and Player #2 is still away from the table.)
- Player #4 decides to swap his present with Player #1. Player #4 takes the money from his newly acquired present and leaves the table. Player #1 opens his new present (which used to belong to Player #4).

The game ends after all the presents are opened, and all players keep the money in their currently held presents.

Suppose each player follows a strategy that maximizes the expected value that the player keeps at the end of the game.

(a) Find, with proof, the strategy each player follows. That is, describe when each player will choose to swap presents with someone, or keep her original present.

(b) What is the expected number of swaps?

(a) At all times, each player knows the set of amounts in the unopened presents. We prove the following:

Lemma: The optimal is strategy is: if the largest opened present in play (that is, available to be swapped) is greater than the amount of at least one unopened present, the player whose turn it is should swap his present for the largest opened present in play. Otherwise, he should open his present. If all players follow this strategy, then any player who opens his own present will end the game with the lowest amount of those unopened at the time of his turn.

Proof of Lemma: We prove by induction, in reverse order of the players' numbers.

Base case: Player #50 will clearly benefit by following the strategy: if there is a present in play that is larger than the final unopened present, then she should swap for the larger amount. Otherwise, the unopened present is the largest amount available to her, so she should open it: the game will immediately end and she will get to keep the amount that she just opened.

Inductive step: Let $1 \leq k < 50$ be an integer, and assume that it is Player $\#k$'s turn, and that (by inductive hypothesis) Players $\#(k+1)$ through #50 will follow the strategy as outlined in the Lemma.

If all the available amounts are less than all the unopened amounts, then Player $\#k$ will do strictly worse by swapping than by opening: swapping is guaranteed to result in a lower amount than opening. So Player $\#k$ should open his present; after this, all subsequent players will swap with Player $\#k$ as long as there is a smaller-valued present remaining unopened, hence Player $\#k$ is guaranteed to be stuck with the smallest remaining unopened amount at the end of the game.

On the other hand, suppose that Player $\#m$, with $m < k$, holds an opened present with an amount M that is larger than some amount(s) that are currently unopened. If Player $\#k$ chooses the opposite strategy to the Lemma's, and opens his present, there are two possibilities:

Case 1: Player $\#k$ opens an amount less than M. Players will only swap with Player $\#k$ if he has the largest opened present and there is an unopened present with smaller value. Once they start swapping with Player $\#k$, they will keep swapping until Player $\#k$ either receives that unopened present or is no longer the largest opened present; in either case Player $\#k$'s value will decrease, and will still be less than M.

Case 2: Player $\#k$ opens an amount greater than M. Then since there is still an unopened present of value less than M, all subsequent players will follow the strategy and swap with Player $\#k$, until Player $\#k$ is stuck with a present with amount less than M, at which point we revert to Case 1.

In any event, Player $\#k$ ends up with a final amount less than M if he does not swap, whereas he ends up with M if he swaps. So he should swap.

This completes the proof. □

(b) Based on the Lemma, there is no swap if and only if all of the opened presents are less than all of the unopened presents. (Note this includes Player #1's non-swap.) So we count the expected number of non-swaps and subtract from n.

Player #1's turn is always a non-swap, and the next non-swap occurs after the player who holds (and swaps away) the present with \$1. The next non-swap occurs after the player holding the minimum value of all those remaining, and so on. This allows us to set up a recurrence.

Let E_k denote the expected number of non-swaps in the k-player version of the game. Note that $E_0 = 0$ and $E_1 = 1$. If the \$1 present is held by person $\#j$, then there are E_{k-j} swaps expected after the initial swap. Therefore, since the \$1 present is equally likely to be held by any player, the recurrence is

$$E_k = 1 + \frac{1}{k}(E_{k-1} + E_{k-2} + \cdots + E_1 + E_0) = 1 + \frac{1}{k}\sum_{j=0}^{k-1} E_j.$$

We prove by induction that

$$E_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = \sum_{j=1}^{k} \frac{1}{j}.$$

It suffices to show that $E_k - E_{k-1} = \frac{1}{k}$ (since $E_1 = 1$) but this follows from:

$$\begin{aligned}
E_k - E_{k-1} &= \left(1 + \frac{1}{k}\sum_{j=0}^{k-1} E_j\right) - \left(1 + \frac{1}{k-1}\sum_{j=0}^{k-2} E_j\right) \\
&= \frac{1}{k}E_{k-1} + \left(\frac{1}{k} - \frac{1}{k-1}\right)\sum_{j=0}^{k-2} E_j \\
&= \frac{1}{k}E_{k-1} - \frac{1}{k(k-1)}\sum_{j=0}^{k-2} E_j \\
&= \frac{1}{k}E_{k-1} - \frac{1}{k}\left(1 + \frac{1}{k-1}\sum_{j=0}^{k-2} E_j - 1\right) \\
&= \frac{1}{k}E_{k-1} - \frac{1}{k}(E_{k-1} - 1) \\
&= \frac{1}{k},
\end{aligned}$$

proving the formula.

Thus, the expected number of swaps in the 50-person game is 50 minus the expected number of non-swaps, which is

$$50 - E_{50} = 50 - \left(1 + \frac{1}{2} + \cdots + \frac{1}{50}\right) = \frac{1410089876350757780359241}{3099044504245996706400} \approx 45.501.$$