USAMTS

USA Mathematical Talent Search

Round 1 Solutions

Year 29 — Academic Year 2017–2018

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1/1/29. Fill each white square in with a number so that each of the 27 three-digit numbers whose digits are all 1, 2, or 3 is used exactly once. For each pair of white squares sharing a side, the two numbers must have equal digits in exactly two of the three positions (ones, tens, hundreds). Some numbers have been given to you.

					212		
	■		213	■		■	123
	■					131	
	■	121	■		113	■	
				312			

111 112 ~~113~~ 211 ~~212~~ ~~213~~ 311 ~~312~~ 313
~~121~~ 122 ~~123~~ 221 222 223 321 322 323
~~131~~ 132 133 231 232 233 331 332 333

Solution

			233	232	212	222	122
231	■	223	213	■	112	■	123
331	■	221	211	311	111	131	133
332	■	121	■	313	113	■	132
333	323	321	322	312			

2/1/29. After each Goober ride, the driver rates the passenger as 1, 2, 3, 4, or 5 stars. The passenger's overall rating is determined as the average of all of the ratings given to him or her by drivers so far. Noah had been on several rides, and his rating was neither 1 nor 5. Then he got a 1 star on a ride because he barfed on the driver. Show that the number of 5 stars that Noah needs in order to climb back to at least his overall rating before barfing is independent of the number of rides that he had taken.

Solution

Suppose Noah's original rating was r. Then, we simply need the average of the 1-star ride and all the subsequent 5-star rides to be greater than or equal to r. This does not depend on how many rides Noah took to get a rating of r. In particular, if we let n be the minimum number of rides Noah needs to take, we know that

$$\frac{1+5n}{n+1} \geq r,$$

or $1 + 5n \geq rn + r$. So,

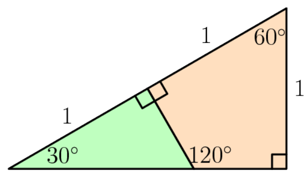
$$n \geq \frac{r-1}{5-r}.$$

Hence, the minimum number of rides Noah has to take is $\boxed{\left\lceil \frac{r-1}{5-r} \right\rceil}$.

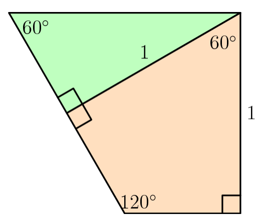
3/1/29. Do there exist two polygons such that, by putting them together in three different ways (without holes, overlap, or reflections), we can obtain first a triangle, then a convex quadrilateral, and lastly a convex pentagon?

Solution

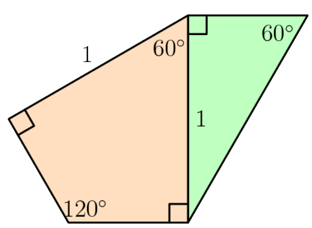
The answer is $\boxed{\text{yes}}$. First, we break the large 30-60-90 triangle into two pieces as shown below, a green triangle and an orange quadrilateral, and we will use these as our two polygons.



Next, we move the green triangle to form a quadrilateral as shown.



Finally, we move the green triangle again to give a convex pentagon. Note that the leg of length 1 matches up exactly with the height of the orange quadrilateral.



Problem proposed by Nikolai Beluhov.

4/1/29. Two players take turns placing an unused number from $\{1, 2, 3, 4, 5, 6, 7, 8\}$ into one of the empty squares in the array to the right. The game ends once all the squares are filled. The first player wins if the product of the numbers in the top row is greater. The second player wins if the product of the numbers in the bottom row is greater. If both players play with perfect strategy, who wins this game?

Solution

The first player wins this game if he plays with perfect strategy.

We start by observing that if either player receives both 1 and 2, they lose. If you receive both 1 and 2, your product is at most $1 \cdot 2 \cdot 7 \cdot 8 = 112$, which is smaller than $3 \cdot 4 \cdot 5 \cdot 6 = 360$. So, each player must avoid having both 1 and 2. Now, we can describe a strategy for the first player.

The first player will always give the smallest available number to the second player.

Following this strategy the game must go as follows:

- The first player places a 1 in the bottom row.
- The second player must respond by placing a 2 in the top row; otherwise he loses.
- The first player places a 3 in the bottom row.

Now the board looks like this:

2			
1	3		

The maximum possible product in the bottom row is now $1 \cdot 3 \cdot 7 \cdot 8 = 168$, which is smaller than $2 \cdot 4 \cdot 5 \cdot 6 = 240$. So, the first player has won the game at this point.

5/1/29. Does there exist a set S consisting of rational numbers with the following property? For every integer n there is a unique nonempty, finite subset of S, whose elements sum to n.

Solution

We claim that the answer is $\boxed{\text{yes}}$.

A general construction is as follows. Enumerate the integers as $s_1, s_2, s_3, \ldots$. Then, we can consider the set S, which consists of all numbers of the form $-\frac{1}{10^k}$, where k is a positive integer, along with all numbers of the form $s_k + \sum_{i=1}^{k} \frac{1}{10^i}$.

Then, to obtain a sum of s_n from elements of S, we simply take

$$\left(s_n + \sum_{i=1}^{n} \frac{1}{10^i}\right) - \frac{1}{10} - \frac{1}{100} - \cdots - \frac{1}{10^n}.$$

To conclude, we simply need to show this is unique. To that end, we will show that no finite sum from S that contains two or more elements of the form $s_k + \sum_{i=1}^{k} \frac{1}{10^i}$ can sum to an integer.

Suppose we have a sum of numbers of the form $s_k + \sum_{i=1}^{k} \frac{1}{10^i}$, with $N < M$ as the two largest indices. Then, our sum must end with

$$\frac{1}{10^N} + \frac{1}{10^N} + \frac{1}{10^{N+1}} + \cdots + \frac{1}{10^M} = \frac{2}{10^N} + \frac{1}{10^{N+1}} + \cdots + \frac{1}{10^M}.$$

Then, we cannot subtract off any set of numbers of the form $\frac{1}{10^k}$ to cancel out $\frac{2}{10^N}$. So, we can never take a sum that involves multiple s_k to get an integer. This means that the representation of each integer above must be unique.

Note 1: It will likely help to think about a specific construction, and decimal representation. Our numbers look like: $-.1, -.01, -.001, \ldots$ and $0.1, 1.11, -.889, 2.1111, \ldots$. In this form, it's much more natural to see that they will uniquely make every integer.

Note 2: Another interesting construction is $1, -2, 4, -8, 16, \ldots$. This sequence can uniquely make every integer *except* zero. Unfortunately, adding anything to get 0 breaks almost everything in this sequence!