USA Mathematical Talent Search

Round 3 Problems

Year 23 — Academic Year 2011–2012

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem other than Problem 1 with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by Tuesday, January 17, 2012, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 17, 2012.

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before January 17.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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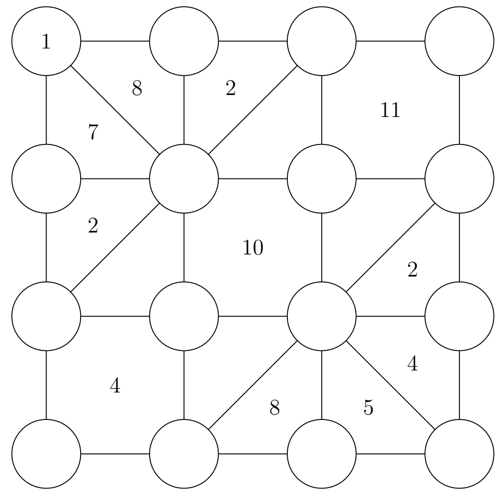
Each problem is worth 5 points.

1/3/23. Fill in the circles to the right with the numbers 1 through 16 so that each number is used once (the number 1 has been filled in already). The number in any non-circular region is equal to the greatest difference between any two numbers in the circles on that region's vertices.

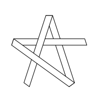
You do not need to prove that your configuration is the only one possible; you merely need to find a valid configuration. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



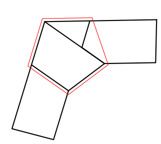
2/3/23. Let x be a complex number such that $x^{2011} = 1$ and $x \neq 1$. Compute the sum

$$\frac{x^2}{x-1} + \frac{x^4}{x^2-1} + \frac{x^6}{x^3-1} + \cdots + \frac{x^{4020}}{x^{2010}-1}.$$

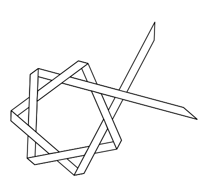
3/3/23. A long, 1-inch wide strip of cloth can be folded into the figure below.



When the cloth is pulled tight and flattened, the result is a knot with two trailing strands. The knot has outer boundary equal to a regular pentagon as shown below.



Instead, a long 1-inch wide strip of cloth is folded into the next figure, following the given turns and crossings.



When the cloth is pulled tight and flattened, the result is a knot with two trailing strands. The knot has outer boundary equal to a regular heptagon. The trailing strands of the heptagonal knot are both cut at the outer (heptagonal) boundary of the knot. Then the knot is untied. What is the area of one side of the resulting quadrilateral of cloth? (Your answer may contain trigonometric expressions.)

4/3/23. Renata the robot packs boxes in a warehouse. Each box is a cube of side length 1 foot. The warehouse floor is a square, 12 feet on each side, and is divided into a 12-by-12 grid of square tiles 1 foot on a side. Each tile can either support one box or be empty. The warehouse has exactly one door, which opens onto one of the corner tiles.

Renata fits on a tile and can roll between tiles that share a side. To access a box, Renata must be able to roll along a path of empty tiles starting at the door and ending at a tile sharing a side with that box.

(a) Show how Renata can pack 91 boxes into the warehouse and still be able to access any box.

(b) Show that Renata **cannot** pack 95 boxes into the warehouse and still be able to access any box.

5/3/23. Let $k > 2$ be a positive integer. Elise and Xavier play a game that has four steps, in this order.

1. Elise picks 2 nonzero digits (1-9), called e and f.
2. Xavier then picks k nonzero digits (1-9), called $x_1, \ldots, x_k$.
3. Elise picks any positive integer d.
4. Xavier picks an integer $b > 10$.

Each player's choices are known to the other player when the choices are made.

The winner is determined as follows. Elise writes down the two-digit base b number ef_b. Next, Xavier writes the k-digit base b number that is constructed by concatenating his digits,

$$(x_1 \ldots x_k)_b.$$

They then compute the greatest common divisor (gcd) of these two numbers. If this gcd is greater than or equal to the integer d then Xavier wins. Otherwise Elise wins.

(As an example game for $k = 3$, Elise chooses the digits $(e, f) = (2, 4)$, Xavier chooses $(4, 4, 8)$, and then Elise picks $d = 100$. Xavier picks base $b = 25$. The base-25 numbers 24_{25} and 448_{25} are, respectively, equal to 54 and 2608. The greatest common divisor of these two is 2, which is much less than 100, so Elise wins handily.)

Find all k for which Xavier can force a win, no matter how Elise plays.