USA Mathematical Talent Search

Round 3 Problems

Year 28 — Academic Year 2016–2017

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by January 3, 2017, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 8 PM Eastern / 5 PM Pacific on January 3, 2017

   (b) Mail: USAMTS
   PO Box 4499
   New York, NY 10163
   (Solutions must be postmarked on or before January 3, 2017.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages.

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/3/28. Fill in each square of the grid with a number from 1 to 16, using each number exactly once. Numbers at the left or top give the largest sum of two numbers in that row or column. Numbers at the right or bottom give the largest difference of two numbers in that row or column.

+	13	28	25	23	
13					7
28					
					8
23					8
	7		12	12	−

You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/28. Malmer Pebane, Fames Jung, and Weven Dare are perfect logicians that always tell the truth. Malmer decides to pose a puzzle to his friends: he tells them that the day of his birthday is at most the number of the month of his birthday. Then Malmer announces that he will whisper the day of his birthday to Fames and the month of his birthday to Weven, and he does exactly that.

After Malmer whispers to both of them, Fames thinks a bit, then says "Weven cannot know what Malmer's birthday is."

After that, Weven thinks a bit, then says "Fames also cannot know what Malmer's birthday is."

This exchange repeats, with Fames and Weven speaking alternately and each saying the other can't know Malmer's birthday. However, at one point, Weven instead announces "Fames and I can now know what Malmer's birthday is. Interestingly, that was the longest conversation like that we could have possibly had before both figuring out Malmer's birthday."

Find Malmer's birthday.

3/3/28. An *n-city* is an $n \times n$ grid of positive integers such that every entry greater than 1 is the sum of an entry in the same row and an entry in the same column. Shown below is an example 3-city.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 6 & 4 & 1 \end{pmatrix}$$

(a) Construct a 5-city that includes some entry that is at least 150. (It is acceptable simply to write the 5-city. You do not need to explain how you found it.)

(b) Show that for all $n \geq 1$, the largest entry in an n-city is at most $3^{\binom{n}{2}}$.

4/3/28. Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be sets of points in the plane. Suppose that for all points x,

$$D(x, A_1) + D(x, A_2) + \cdots + D(x, A_n) \geq D(x, B_1) + D(x, B_2) + \cdots + D(x, B_n),$$

where $D(x, y)$ denotes the distance between x and y. Show that the A_i's and the B_i's share the same center of mass.

5/3/28. Consider the set $S = \{q + \frac{1}{q},$ where q ranges over all positive rational numbers$\}$.

(a) Let N be a positive integer. Show that N is the sum of two elements of S if and only if N is the product of two elements of S.

(b) Show that there exist infinitely many positive integers N that cannot be written as the sum of two elements of S.

(c) Show that there exist infinitely many positive integers N that can be written as the sum of two elements of S.

Problems by Billy Swartworth, Remus Nicoara, and USAMTS Staff.
Round 3 Solutions must be submitted by **January 3, 2017**.
Please visit **http://www.usamts.org** for details about solution submission.