USA Mathematical Talent Search

Round 3 Problems

Year 16 — Academic Year 2004–2005

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1. Put your name, username, and USAMTS ID# on **every page you submit**.
2. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
3. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.
4. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.
5. Do not fax solutions written in pencil.
6. No single page should contain solutions to more than one problem.
7. By the end of December, Round 2 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS.
8. Submit your solutions by January 3, 2005 (postmark deadline), via one of the methods below.
   (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.
   (b) Fax: (619) 445-2379
   (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
9. Re–read item 1.

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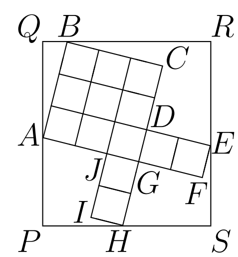
1/3/16. Given two integers x and y, let $(x\|y)$ denote the *concatenation* of x by y, which is obtained by appending the digits of y onto the end of x. For example, if $x = 218$ and $y = 392$, then $(x\|y) = 218392$.
(a) Find 3-digit integers x and y such that $6(x\|y) = (y\|x)$.
(b) Find 9-digit integers x and y such that $6(x\|y) = (y\|x)$.

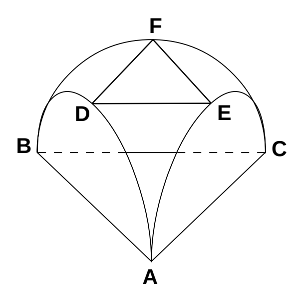
2/3/16. Find three isosceles triangles, no two of which are congruent, with integer sides, such that each triangle's area is numerically equal to 6 times its perimeter.

3/3/16. Define the recursive sequence 1, 4, 13, ... by $s_1 = 1$ and $s_{n+1} = 3s_n + 1$ for all positive integers n. The element $s_{18} = 193710244$ ends in two identical digits. Prove that all the elements in the sequence that end in two or more identical digits come in groups of three consecutive elements that have the same number of identical digits at the end.

4/3/16. Region $ABCDEFGHIJ$ consists of 13 equal squares and is inscribed in rectangle $PQRS$ with A on $\overline{PQ}$, B on $\overline{QR}$, E on $\overline{RS}$, and H on $\overline{SP}$, as shown in the figure on the right. Given that $PQ = 28$ and $QR = 26$, determine, with proof, the area of region $ABCDEFGHIJ$.



5/3/16. Consider an isosceles triangle ABC with side lengths $AB = AC = 10\sqrt{2}$ and $BC = 10\sqrt{3}$. Construct semicircles P, Q, and R with diameters AB, AC, BC respectively, such that the plane of each semicircle is perpendicular to the plane of ABC, and all semicircles are on the same side of plane ABC as shown. There exists a plane above triangle ABC that is tangent to all three semicircles P, Q, R at the points D, E, and F respectively, as shown in the diagram. Calculate, with proof, the area of triangle DEF.



Round 3 Solutions must be submitted by **January 3, 2005**.
Please visit **http://www.usamts.org** for details about solution submission.