USA Mathematical Talent Search

Round 1 Problems

Year 17 — Academic Year 2005–2006

www.usamts.org

Please follow the rules below to ensure that your paper is graded properly.

1. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.

2. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.

3. Put your name and USAMTS ID# on **every page you submit**.

4. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into the site, then clicking on My USAMTS on the sidebar, then click Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.

6. Do not fax solutions written in pencil.

7. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

8. By the end of October, Round 1 results will be posted at www.usamts.org. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.

9. Submit your solutions by October 3, 2005 (postmark deadline), via one (and only one!) of the methods below.

   (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send .doc Microsoft Word files.

   (b) Fax: (619) 445-2379 (Please include a cover sheet indicating the number of pages you are faxing, your name, and your User ID.)

   (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.

10. Re–read Items 1–9.

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1/1/17. An increasing arithmetic sequence with infinitely many terms is determined as follows. A single die is thrown and the number that appears is taken as the first term. The die is thrown again and the second number that appears is taken as the common difference between each pair of consecutive terms. Determine with proof how many of the 36 possible sequences formed in this way contain at least one perfect square.

2/1/17. George has six ropes. He chooses two of the twelve loose ends at random (possibly from the same rope), and ties them together, leaving ten loose ends. He again chooses two loose ends at random and joins them, and so on, until there are no loose ends. Find, with proof, the expected value of the number of loops George ends up with.

3/1/17. Let r be a nonzero real number. The values of z which satisfy the equation

$$r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) = 0$$

are plotted on the complex plane (i.e. using the real part of each root as the x-coordinate and the imaginary part as the y-coordinate). Show that the area of the convex quadrilateral with these points as vertices is independent of r, and find this area.

4/1/17. Homer gives mathematicians Patty and Selma each a different integer, not known to the other or to you. Homer tells them, within each other's hearing, that the number given to Patty is the product ab of the positive integers a and b, and that the number given to Selma is the sum $a + b$ of the same numbers a and b, where $b > a > 1$. He doesn't, however, tell Patty or Selma the numbers a and b. The following (honest) conversation then takes place:

Patty: "I can't tell what numbers a and b are."
Selma: "I knew before that you couldn't tell."
Patty: "In that case, I now know what a and b are."
Selma: "Now I also know what a and b are."

Supposing that Homer tells *you* (but neither Patty nor Selma) that neither a nor b is greater than 20, find a and b, and prove your answer can result in the conversation above.

5/1/17. Given triangle ABC, let M be the midpoint of side $\overline{AB}$ and N be the midpoint of side $\overline{AC}$. A circle is inscribed inside quadrilateral $NMBC$, tangent to all four sides, and that circle touches $\overline{MN}$ at point X. The circle inscribed in triangle AMN touches $\overline{MN}$ at point Y, with Y between X and N. If $XY = 1$ and $BC = 12$, find, with proof, the lengths of the sides $\overline{AB}$ and $\overline{AC}$.

Round 1 Solutions must be submitted by **October 3, 2005**.
Please visit **http://www.usamts.org** for details about solution submission.