# USA Mathematical Talent Search

Round 3 Problems

Year 19 — Academic Year 2007–2008

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**Please follow the rules below:**

1. You must show your work and prove your answers on all problems. If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. If you have not already sent an Entry Form, download an Entry Form from the Forms page at

   `http://www.usamts.org/MyUSAMTS/U_MyForms.php`

   and submit the completed form with your solutions.

3. If you have already sent in an Entry Form and a Permission Form, you do not need to resend them.

4. Put your name and USAMTS ID# on **every page you submit**.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging into www.usamts.org, then clicking on My USAMTS on the sidebar, then clicking Profile. If you are registered for the USAMTS and haven't received any email from us about the USAMTS, your email address is probably wrong in your Profile.

7. Do not fax solutions written in pencil.

8. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

9. Round 3 results will be posted at www.usamts.org by mid-February. To see your results, log in to the USAMTS page, then go to My USAMTS. Check that your email address in your USAMTS Profile is correct; you will receive an email when the scores are available.

10. Submit your solutions by January 8, 2008 (postmark deadline), via one (and only one!) of the methods below.

    (a) Email: solutions@usamts.org. Please see usamts.org for a list of acceptable file types. Do not send Microsoft Word files.

    (b) Fax: (619) 445-2379 (You must include a cover sheet indicating the number of pages you are faxing, your name, and your USAMTS ID#.)

    (c) Snail mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.

11. Re–read Items 1–10.

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**1/3/19.** We construct a sculpture consisting of infinitely many cubes, as follows. Start with a cube with side length 1. Then, at the center of each face, attach a cube with side length $\frac{1}{3}$ (so that the center of a face of each attached cube is the center of a face of the original cube). Continue this procedure indefinitely: at the center of each exposed face of a cube in the structure, attach (in the same fashion) a smaller cube with side length one-third that of the exposed face. What is the volume of the entire sculpture?

**2/3/19.** Gene starts with the $3 \times 3$ grid of 0's shown at left below. He then repeatedly chooses a $2 \times 2$ square within the grid and increases all four numbers in the chosen $2 \times 2$ square by 1. One possibility for Gene's first three steps is shown below.

$$\begin{array}{|c|c|c|}\hline 0&0&0\\\hline 0&0&0\\\hline 0&0&0\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 0&0&0\\\hline 1&1&0\\\hline 1&1&0\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 0&1&1\\\hline 1&2&1\\\hline 1&1&0\\\hline\end{array} \to \begin{array}{|c|c|c|}\hline 0&1&1\\\hline 2&3&1\\\hline 2&2&0\\\hline\end{array}$$

How many different grids can be produced with this method such that each box contains an integer from 1 to 12, inclusive? (The numbers in the boxes need not be distinct.)

**3/3/19.** Consider all polynomials $f(x)$ with integer coefficients such that $f(200) = f(7) = 2007$ and $0 < f(0) < 2007$. Show that the value of $f(0)$ does not depend on the choice of polynomial, and find $f(0)$.

**4/3/19.** Prove that 101 divides infinitely many of the numbers in the set

$$\{2007, 20072007, 200720072007, 2007200720072007, \ldots\}.$$

**5/3/19.** For every rational number $0 < \dfrac{p}{q} < 1$, where $p$ and $q$ are relatively prime, construct a circle with center $\left(\dfrac{p}{q}, \dfrac{1}{2q^2}\right)$ and diameter $\dfrac{1}{q^2}$. Also construct circles centered at $\left(0, \dfrac{1}{2}\right)$ and $\left(1, \dfrac{1}{2}\right)$ with diameter 1.

(a) Prove that any two such circles intersect in at most 1 point.

(b) Prove that the total area of all of the circles is $\dfrac{\pi}{4}\left(1 + \dfrac{\sum_{i=1}^{\infty} \frac{1}{i^3}}{\sum_{i=1}^{\infty} \frac{1}{i^4}}\right)$.

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Round 3 Solutions must be submitted by **January 8, 2008**.

Please visit **http://www.usamts.org** for details about solution submission.