USA Mathematical Talent Search

Round 3 Problems

Year 20 — Academic Year 2008–2009

www.usamts.org

Important! New deadline for web submission: 3 PM Eastern / Noon Pacific

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by January 12, 2009, via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 12
   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before the deadline day.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)
7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

USA Mathematical Talent Search

Round 3 Problems

Year 20 — Academic Year 2008–2009

www.usamts.org

Each problem is worth 5 points.

1/3/20. Let S be the set of all 10-digit numbers (which by definition may not begin with 0) in which each digit 0 through 9 appears exactly once. For example, the number 3,820,956,714 is in S. A number n is picked from S at random. What is the probability that n is divisible by 11?

2/3/20. Two players are playing a game that starts with 2009 stones. The players take turns removing stones. A player may remove exactly 3, 4, or 7 stones on his or her turn, except that if only 1 or 2 stones are remaining then the player may remove them all. The player who removes the last stone wins. Determine, with proof, which player has a winning strategy, the first or the second player.

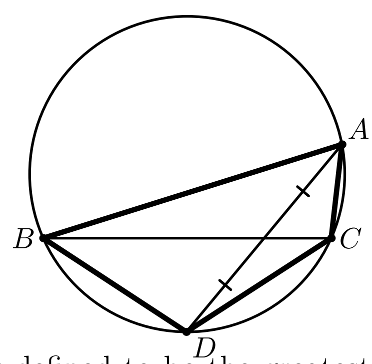
3/3/20. Let a, b, c be three positive integers such that

$$(\operatorname{lcm}(a,b))(\operatorname{lcm}(b,c))(\operatorname{lcm}(c,a)) = (abc)\gcd(a,b,c),$$

where "lcm" means "least common multiple" and "gcd" means "greatest common divisor." Given that no quotient of any two of a, b, c is an integer (that is, none of a, b, c is an integer multiple of any other of a, b, c), find the minimum possible value of $a + b + c$.

4/3/20. Given a segment $\overline{BC}$ in plane $\mathcal{P}$, find the locus of all points A in $\mathcal{P}$ with the following property:

There exists *exactly one* point D in $\mathcal{P}$ such that $ABDC$ is a cyclic quadrilateral and $\overline{BC}$ bisects $\overline{AD}$, as shown at right.



5/3/20. Let b be an integer such that $b \geq 2$, and let $a > 0$ be a real number such that $\dfrac{1}{a} + \dfrac{1}{b} > 1$. Prove that the sequence

$$\lfloor a \rfloor, \lfloor 2a \rfloor, \lfloor 3a \rfloor, \ldots$$

contains infinitely many integral powers of b. (Note that $\lfloor x \rfloor$ is defined to be the greatest integer less than or equal to x.)

Round 3 Solutions must be submitted by **January 12, 2009**.

Web deadline: 3 PM Eastern / Noon Pacific on the due date

Mail deadline: Postmarked on or before due date

Please visit **http://www.usamts.org** for details about solution submission.