USA Mathematical Talent Search

Round 1 Problems

Year 21 — Academic Year 2009–2010

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by October 13, 2009, via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on October 13

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before October 13.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto `www.usamts.org` and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)

7. Round 1 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on `www.usamts.org`.

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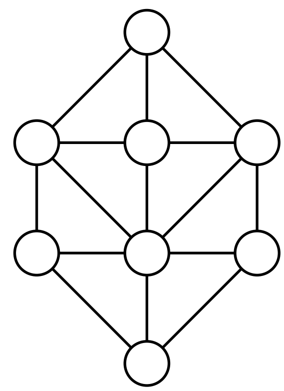
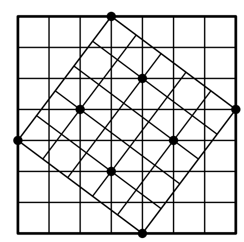
Each problem is worth 5 points.

1/1/21. Fill in the circles in the picture at right with the digits 1-8, one digit in each circle with no digit repeated, so that no two circles that are connected by a line segment contain consecutive digits. In how many ways can this be done?



2/1/21. The ordered pair of four-digit numbers $(2025, 3136)$ has the property that each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number. Find, with proof, all ordered pairs of five-digit numbers and ordered pairs of six-digit numbers with the same property: each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number.

3/1/21. A square of side length 5 is inscribed in a square of side length 7. If we construct a grid of 1×1 squares for both squares, as shown to the right, then we find that the two grids have 8 lattice points in common. If we do the same construction by inscribing a square of side length 1489 in a square of side length 2009, and construct a grid of 1×1 squares in each large square, then how many lattice points will the two grids of 1×1 squares have in common?



4/1/21. Let $ABCDEF$ be a convex hexagon, such that $FA = AB$, $BC = CD$, $DE = EF$, and $\angle FAB = 2\angle EAC$. Suppose that the area of ABC is 25, the area of CDE is 10, the area of EFA is 25, and the area of ACE is x. Find, with proof, all possible values of x.

5/1/21. The cubic equation $x^3+2x-1 = 0$ has exactly one real root r. Note that $0.4 < r < 0.5$.

(a) Find, with proof, an increasing sequence of positive integers $a_1 < a_2 < a_3 < \cdots$ such that

$$\frac{1}{2} = r^{a_1} + r^{a_2} + r^{a_3} + \cdots .$$

(b) Prove that the sequence that you found in part (a) is the unique increasing sequence with the above property.

Round 1 Solutions must be submitted by **October 13, 2009**.
Please visit **http://www.usamts.org** for details about solution submission.