USA Mathematical Talent Search

Round 3 Problems

Year 21 — Academic Year 2009–2010

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.

2. Put your name and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.

4. Submit your solutions by January 11, 2010, via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on January 11

   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090.
   (Solutions must be postmarked on or before January 11.)

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)

7. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/3/21. Let $ABCD$ be a convex quadrilateral with $AC \perp BD$, and let P be the intersection of AC and BD. Suppose that the distance from P to AB is 99, the distance from P to BC is 63, and the distance from P to CD is 77. What is the distance from P to AD?

2/3/21. Find, with proof, a positive integer n such that

$$\frac{(n+1)(n+2)\cdots(n+500)}{500!}$$

is an integer with no prime factors less than 500.

3/3/21. We are given a rectangular piece of white paper with length 25 and width 20. On the paper we color blue the interiors of 120 disjoint squares of side length 1 (the sides of the squares do not necessarily have to be parallel to the sides of the paper). Prove that we can draw a circle of diameter 1 on the remaining paper such that the entire interior of the circle is white.

4/3/21. Let a and b be positive integers such that all but 2009 positive integers are expressible in the form $ma + nb$, where m and n are nonnegative integers. If 1776 is one of the numbers that is not expressible, find $a + b$.

5/3/21. The sequences (a_n), (b_n), and (c_n) are defined by $a_0 = 1$, $b_0 = 0$, $c_0 = 0$, and

$$a_n = a_{n-1} + \frac{c_{n-1}}{n}, \quad b_n = b_{n-1} + \frac{a_{n-1}}{n}, \quad c_n = c_{n-1} + \frac{b_{n-1}}{n}$$

for all $n \geq 1$. Prove that

$$\left| a_n - \frac{n+1}{3} \right| < \frac{2}{\sqrt{3n}}$$

for all $n \geq 1$.

Round 3 Solutions must be submitted by **January 11, 2010**.

Please visit **http://www.usamts.org** for details about solution submission.