USA Mathematical Talent Search

Round 4 Problems

Year 21 — Academic Year 2009–2010

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer for a problem with no proof, you will get no more than 1 point.
2. Put your name and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page, and you should only submit work on one side of each piece of paper.
4. Submit your solutions by March 8, 2010, via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 3 PM Eastern / Noon Pacific on March 8
   (b) Mail: USAMTS, P.O. Box 2090, Alpine, CA 91903–2090. (Solutions must be postmarked on or before March 8.)
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging onto www.usamts.org and visiting the "My USAMTS" pages. (If you are registered for the USAMTS and haven't received any email from us about the USAMTS, then your email address is probably wrong in your Profile.)
7. Round 4 and full-year results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules on www.usamts.org.

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Each problem is worth 5 points.

1/4/21. Archimedes planned to count all of the prime numbers between 2 and 1000 using the Sieve of Eratosthenes as follows:

(a) List the integers from 2 to 1000.

(b) Circle the smallest number in the list and call this p.

(c) Cross out all multiples of p in the list except for p itself.

(d) Let p be the smallest number remaining that is neither circled nor crossed out. Circle p.

(e) Repeat steps (c) and (d) until each number is either circled or crossed out.

At the end of this process, the circled numbers are prime and the crossed out numbers are composite.

Unfortunately, while crossing off the multiples of 2, Archimedes accidentally crossed out two odd primes in addition to crossing out all the even numbers (besides 2). Otherwise, he executed the algorithm correctly. If the number of circled numbers remaining when Archimedes finished equals the number of primes from 2 to 1000 (including 2), then what is the largest possible prime that Archimedes accidentally crossed out?

2/4/21. Let a, b, c, d be four real numbers such that

$$a + b + c + d = 8,$$
$$ab + ac + ad + bc + bd + cd = 12.$$

Find the greatest possible value of d.

3/4/21. I give you a deck of n cards numbered 1 through n. On each turn, you take the top card of the deck and place it anywhere you choose in the deck. You must arrange the cards in numerical order, with card 1 on top and card n on the bottom. If I place the deck in a random order before giving it to you, and you know the initial order of the cards, what is the expected value of the minimum number of turns you need to arrange the deck in order?

4/4/21. Let S be a set of 10 distinct positive real numbers. Show that there exist $x, y \in S$ such that

$$0 < x - y < \frac{(1 + x)(1 + y)}{9}.$$

5/4/21. Tina and Paul are playing a game on a square $\mathcal{S}$. First, Tina selects a point T inside $\mathcal{S}$. Next, Paul selects a point P inside $\mathcal{S}$. Paul then colors blue all the points inside $\mathcal{S}$ that are closer to P than T. Tina wins if the blue region thus produced is the interior of a triangle. Assuming that Paul is lazy and simply selects his point at random (and that Tina knows this), find, with proof, a point Tina can select to maximize her probability of winning, and compute this probability.

Round 4 Solutions must be submitted by **March 8, 2010**.

Please visit **http://www.usamts.org** for details about solution submission.