

# U S A Mathematical Talent Search

## PROBLEMS / SOLUTIONS / COMMENTS

### Round 2 - Year 10 - Academic Year 1998-99

Gene A. Berg, Editor

**1/2/10.** Determine the unique pair of real numbers  $(x, y)$  that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28.$$

**Solution 1 by Robert Kotredes (11/ME):** The polynomial  $4x^2 + 6x + 4$  can be written as

$4\left(x + \frac{3}{4}\right)^2 + \frac{7}{4}$ , and therefore has a range of  $\geq \frac{7}{4}$ . The polynomial  $4y^2 - 12y + 25$  can be written as  $4\left(x - \frac{3}{2}\right)^2 + 16$ , and therefore has a range of  $\geq 16$ . Because  $\frac{7}{4} \cdot 16 = 28$ , the only possible values for each polynomial are their minimums, which occur at  $x = -\frac{3}{4}$  and  $y = \frac{3}{2}$ ,

respectively. So the unique pair of real numbers  $(x, y)$  is  $\left(-\frac{3}{4}, \frac{3}{2}\right)$ .

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**Solution 2 by Kim Won Jong (12/CA):** Let  $A = (4x^2 + 6x + 4)$ , then

$$A \cdot (4y^2 - 12y + 25) = 28$$

$$(4y^2 - 12y + 25) = \frac{28}{A}$$

$$4y^2 - 12y + 25 - \frac{28}{A} = 0$$

Since there is a unique pair  $(x, y)$ , the discriminant  $b^2 - 4ac$  of the quadratic formula must equal zero, or

$$12^2 - 4 \cdot 4 \cdot \left(25 - \frac{28}{A}\right) = 0$$

$$144 - 400 + \frac{448}{A} = 0$$

$$A = \frac{7}{4}$$

So,

$$A = 4x^2 + 6x + 4 = \frac{7}{4}$$

$$4x^2 + 6x + \frac{9}{4} = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} = -\frac{3}{4}, \text{ since again, the discriminant is zero.}$$

Substituting A into the original equation,

$$\frac{7}{4}(4y^2 - 12y + 25) = 28$$

$$4y^2 - 12y + 9 = 0$$

$$y = -\frac{b}{2a} = \frac{12}{2 \cdot 4} = \frac{3}{2}$$

Therefore, the unique pair of real numbers  $(x, y)$  that satisfy the equation

$$(4x^2 + 6x + 4)(4y^2 - 12y + 25) = 28$$

is  $\left(-\frac{3}{4}, \frac{3}{2}\right)$ .

**Solution 3 by Robert Klein (12/PA):** “Completing the square” within each term we get:

$$\left(\left(4x^2 + 6x + \frac{9}{4}\right) + \frac{7}{4}\right)\left((4y^2 - 12y + 9) + 16\right) = \left(\left(2x + \frac{3}{2}\right)^2 + \frac{7}{4}\right)\left((2y - 3)^2 + 16\right) = 28$$

Substituting  $a = \left(2x + \frac{3}{2}\right)^2$  and  $b = (2y - 3)^2$ , the equation becomes  $\left(a + \frac{7}{4}\right)(b + 16) = 28$ .

Observing  $a \geq 0$  and  $b \geq 0$ , we must have  $a = 0$  and  $b = 0$  since  $\frac{7}{4} \cdot 16 = 28$ .

Solving  $a = \left(2x + \frac{3}{2}\right)^2 = 0$  and  $b = (2y - 3)^2 = 0$ , the unique pair of real numbers satisfying the equation is  $(-3/4, 3/2)$ .

**Solution 4 by Adam Salem (12/NY):** This solution uses calculus. Let  $f(x) = 4x^2 + 6x + 4$  and  $g(y) = 4y^2 - 12y + 25$ . Thus  $f(x) \cdot g(y) = 28$ . To find the extreme points of both  $f$  and  $g$ , set their derivatives equal to 0.  $f'(x) = 8x + 6$  and  $f''(x) = 8 > 0$ , so  $f(x)$  has a minimum at  $x = -0.75$ . Similarly,  $g'(y) = 8y - 12$  and  $g''(y) = 8 > 0$ , so  $g(y)$  has a minimum at  $y = 1.5$ . This is equivalent to  $f(x) \geq f(-0.75) = 1.75$  and  $g(y) \geq g(1.5) = 16$ . Multiplying the inequalities yields  $f(x)g(y) \geq 28$ . This is notable because 28 is the number from the original equation, which means that we are simply trying to find numbers  $x$  and  $y$  that minimize  $f(x) \cdot g(y)$ . Since both  $f(x)$  and  $g(y)$  are always positive, it follows that the smallest value of  $f(x) \cdot g(y)$  is the product of their minimums. Thus the smallest value of  $f(x) \cdot g(y)$  is produced when

$$(x, y) = (-0.75, 1.5)$$

**Editor's Comment:** This problem is based on a similar problem (E: 11418) proposed by Petre Bătrânetu of Galati, Romania, in Issue 7-8/1997 of the *Gazeta Mathematică*.

**2/2/10.** Prove that there are infinitely many ordered triples of positive integers  $(a, b, c)$  such that the greatest common divisor of  $a, b,$  and  $c$  is 1, and the sum  $a^2b^2 + b^2c^2 + c^2a^2$  is the square of an integer.

**Solution 1 by Irena Foygel (10/IL):** Let  $x$  and  $y$  be relatively prime positive integers such that  $x \equiv 1 \pmod{2}$ . Let  $a = x^2$ ,  $b = 2y^2$ , and  $c = xy$ . Because  $x$  and  $y$  are relatively prime and  $x$  is not divisible by 2,  $a$  and  $b$  are relatively prime; therefore  $\gcd(a, b, c) = 1$ .

Now,

$$\begin{aligned} a^2b^2 + b^2c^2 + c^2a^2 &= (x^4)(4y^4) + (4y^4)(x^2y^2) + (x^2y^2)(x^4) \\ &= x^2y^2(4x^2y^2 + 4y^4 + x^4) = (xy(x^2 + 2y^2))^2 \end{aligned}$$

Set  $n = xy(x^2 + 2y^2)$  and observe  $a^2b^2 + b^2c^2 + c^2a^2 = n^2$ .

Because there are an infinite number of pairs  $(x, y)$  meeting the above requirements, there are an infinite number of triples  $(a, b, c)$  meeting the requirements.

**Solution 2 by Michael Castleman (12/MA):** For relatively prime integers  $a, b,$  and  $c$ , the sum  $a^2b^2 + b^2c^2 + c^2a^2$  is a square of an integer if one of the numbers equals the sum of the other two. We shall now prove this.

Without losing the generality of the proof, assume that  $a + b = c$ . Replacing  $a + b$  for  $c$  and simplifying, we get:

$$\begin{aligned} a^2b^2 + b^2(a+b)^2 + (a+b)^2a^2 &= a^2b^2 + (a^2 + b^2)(a+b)^2 \\ &= a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4 \\ &= (a^2 + ab + b^2)^2 \end{aligned}$$

Since  $a^2 + ab + b^2$  is an integer, the result is the square of an integer. Since there are an infinite number of ordered triples  $(a, b, c)$  such that  $a, b,$  and  $c$  are relatively prime and  $a + b = c$ , and, for all of those pairs,  $a^2b^2 + b^2c^2 + c^2a^2$  is an integer, there exist an infinite number of ordered triples which meet the given criteria.

**Solution 3 by Andy Large (11/TN):** In order for  $a^2b^2 + b^2c^2 + c^2a^2$  to be the square of an integer, it must be possible to write it in the form  $x^2 + 2xy + y^2$  or  $(x + y)^2$ . Conveniently, we have a trinomial with squared first and third terms:  $(ab)^2$  and  $(ca)^2$ . This means we only need the middle term to be equivalent to 2 times the product of the said squared terms. That is

$$b^2c^2 = 2(ab)(ac) = 2a^2bc$$

$$\frac{bc}{2} = a^2$$

$$a = \sqrt{\frac{bc}{2}}$$

$a$  will be an integer under either of the conditions:

(i)  $b = 2^{2n+1} \cdot m^2$  and  $c = s^2$  for  $m, n, s$  integers.

$$\text{Here } a = \sqrt{\frac{(2^{2n+1} \cdot m^2)s^2}{2}} = 2^n \cdot ms$$

(ii)  $c$  and  $b$  are switched in (i) above.

Consider the triple  $(a, b, c) = (2^n ms, 2^{2n+1} m^2, s^2)$ , where  $m, n$ , and  $s$  are positive integers. When  $m$  and  $s$  are relatively prime and when  $s$  and  $2$  are relatively prime, then  $\text{GCD}(a, b, c) = 1$ , and this triple meets the requirements above. There are an infinite number of such triples.

**Solution 4 by Jeffrey Palmer (12/NY):** Integers  $a, b$ , and  $c$  have  $\text{GCD} = 1$  if two of the members are distinct primes.

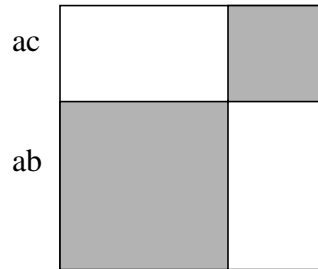
$a^2 b^2 + b^2 c^2 + c^2 a^2$  is the square of an integer when a square can be completed.

The  $a^2 c^2$  and  $a^2 b^2$  terms are accounted by the shaded areas (see diagram at right).

If

$$b^2 c^2 = 2cba^2$$

$$c = \frac{2a^2}{b}$$



If  $b = 2$  and  $a$  is an odd prime, then infinitely many tri-

ples can be created which meet the requirements:  $(a, b, c) = (a, 2, a^2)$ .

**Solution 5 by George Lee (11/CA):** Suppose  $a^2 b^2 + b^2 c^2 + c^2 a^2 = d^2$ , and rewrite the equation as  $d^2 - (a^2 + b^2)c^2 = a^2 b^2$ . With  $a = 1$ , this reduces to  $d^2 - (b^2 + 1)c^2 = b^2$ .

Now let  $b = 1$  to obtain  $d^2 - 2c^2 = 1$ . This is a Pell's equation, which has infinitely many solutions - each corresponding to an ordered triple  $(1, 1, c)$ . [For a discussion of Pell's equation see the Editor's comment following Solution 6.] For example,  $(c, d) = (c_1, d_1) = (2, 3)$  satisfies the equation. Some ways that successive solutions can be generated are by:

(i) using the recursive relations  $c_{n+1} = 3c_n + 2d_n$  and  $d_{n+1} = 4c_n + 3d_n$ ;

(ii) using the single recursive relation  $c_{n+2} = 6c_{n+1} - c_n$  where  $c_1 = 2$  and  $c_2 = 12$ ; or

(iii) using the equation  $c_n = ((3 + 2\sqrt{2})^n - (3 - 2\sqrt{2})^n) / (2\sqrt{2})$ .

For this solution, we will show that the first method works by induction:

$$\begin{aligned}
 d_{n+1}^2 - 2c_{n+1}^2 &= (4c_n + 3d_n)^2 - 2(3c_n + 2d_n)^2 \\
 &= 16c_n^2 + 24c_n d_n + 9d_n^2 - 18c_n^2 - 24c_n d_n - 8d_n^2 \\
 &= d_n^2 - 2c_n^2 \\
 &= 1.
 \end{aligned}$$

Since  $c_{n+1}$  and  $d_{n+1}$  are larger than  $c_n$  and  $d_n$ , we can generate infinitely many solutions  $(c, d) = (c_n, d_n)$ . Also, 1, 1, and  $c_n$  are relatively prime. Thus we can generate infinitely many corresponding solutions  $(1, 1, c)$ .

**Solution 6 by David Fithian (11/OR):** We are to show that infinitely many positive integer sets  $(a, b, c, n)$  satisfy the equation  $a^2 b^2 + b^2 c^2 + c^2 a^2 = n^2$  with  $\gcd(a, b, c) = 1$ . Without loss of generalization, let  $c = 1$ . This automatically makes  $\gcd(a, b, c) = 1$ , regardless of the values of  $a$  and  $b$ . We are left with  $a^2 b^2 + a^2 + b^2 = n^2$  with  $a, b, n \in N$ . Now, set  $b = 1$ , so that  $2a^2 + 1 = n^2$ . Rearranging, we see that  $n^2 - 2a^2 = 1$ ; since this is a Fermat-Pell equation of the form  $x^2 - dy^2 = 1$ , and since  $d = 2$  is prime, there are infinitely many solution pairs  $(a, n)$ , and thus the equation has infinitely many positive integral solutions with  $\gcd(a, b, c)$  equal to 1.

These solutions would be  $(a, b, c, n) = (r, 1, 1, s)$ , where  $s/r$  is a certain fractional convergent of  $\sqrt{2}$ . In particular, the first solution pairs  $(a, n)$  are  $(2, 3), (12, 17), (70, 99), (408, 577), \dots$ . It can be verified that  $n/a$  indeed converges to  $\sqrt{2}$ .

**Editor's Comments:** This problem is due to Suresh T. Thaker of Bombay, India. We are grateful for his contribution. A brief discussion of Pell's equations is available in the solutions to Round 1 of Year 10. Mr. Lee (Solution 5 above) presents two more methods to generate solutions using Pell's equations which we have not included. Pell's equations are used in Solution 6 as well. In the following note, Erin Schram continues the discussion.

**Solution by Pell's Equation,** summary by Erin J. Schram, longtime grader.

Several test-takers solved problem 2 by Pell's equation. Gene Berg described Pell's equation in the solutions to round 1, but he did not expect Pell's equation to be of any use in this round. The creativity of the students who take the USAMTS is surprising and refreshing.

Pell's equation is the quadratic Diophantine equation of the form

$$x^2 - Dy^2 = N$$

where  $D$  and  $N$  are integer constants and  $x$  and  $y$  are integer unknowns. We start with the equation

$$a^2 b^2 + a^2 c^2 + b^2 c^2 = x^2$$

and we pick arbitrary integers for  $a$  and  $b$ . This changes the equation to a Pell's equation in the variables  $x$  and  $c$ . Although setting  $a$  and  $b$  to arbitrary values won't give every solution to  $a^2 b^2 + a^2 c^2 + b^2 c^2 = x^2$ , the problem asked for only an infinite family of solutions, and the solutions to the Pell's equation are an infinite family. Besides, with  $a$  and  $b$  held constant, we can pick them so that their greatest common divisor is 1, forcing the greatest common divisor of  $a$ ,  $b$ , and  $c$  to be 1.

For example, the most common choice was  $a = 1$  and  $b = 1$ . Then the equation simplifies to

$$x^2 - 2c^2 = 1,$$

which is a Pell's equation. This equation is almost identical to the equation from Gene Berg's discussion,  $x^2 - 2y^2 = \pm 1$ . Gene Berg's use of continued fractions to solve that Pell's equation can be adapted to solve problem 2, since every other term from the solutions to  $x^2 - 2y^2 = \pm 1$  satisfies  $x^2 - 2y^2 = 1$ .

For variety, I will discuss the recursive solution to the Pell's equation that results from setting  $a = 1$  and  $b = 2$ , the second most common choice. This gives  $x^2 - 5c^2 = 4$ .

Suppose we have the Pell's equation  $x^2 - Dy^2 = N$  and one solution to it,  $(x_0, y_0)$ . Furthermore, suppose  $(u, v)$  is a solution to  $u^2 - Dv^2 = 1$ : note that I replaced the integer  $N$  with the integer 1 in that equation. Then  $(ux_0 + vDy_0, vx_0 + uy_0)$  will be a solution to  $x^2 - Dy^2 = N$ .

We have  $x^2 - 5c^2 = 4$  and some trial and error gives us  $(x, c) = (3, 1)$  as one solution. For a solution to  $u^2 - Dv^2 = 1$ , let's be lazy and cut our solution to the previous equation in half, which gives  $(u, v) = (1.5, 0.5)$ . Even though the recursion will not be built from integers, it is possible that its results will still be integers. The result is a solution  $(x, c) = (3, 1)$  and a recursive relation:

$$(x_{i+1}, c_{i+1}) = (1.5x_i + 2.5c_i, 0.5x_i + 1.5c_i)$$

Hence, we can rearrange the recursive relation to the following:

$$(x_0, c_0) = (3, 1)$$

$$(x_1, c_1) = (1.5 \times 3 + 2.5 \times 1, 0.5 \times 3 + 1.5 \times 1) = (7, 3)$$

$$(x_2, c_2) = (1.5 \times 7 + 2.5 \times 3, 0.5 \times 7 + 1.5 \times 3) = (18, 8)$$

$$(x_3, c_3) = (1.5 \times 18 + 2.5 \times 8, 0.5 \times 18 + 1.5 \times 8) = (47, 21)$$

$$(x_4, c_4) = (1.5 \times 47 + 2.5 \times 21, 0.5 \times 47 + 1.5 \times 21) = (123, 55)$$

We also have the following relations:

$$c_i = 0.5x_{i-1} + 1.5c_{i-1}, \text{ so } x_{i-1} = 2c_i - 3c_{i-1}$$

$$x_i = 1.5x_{i-1} + 2.5c_{i-1}, \text{ so } x_i = 1.5(2c_i - 3c_{i-1}) + 2.5c_{i-1} = 3c_i - 2c_{i-1}$$

$$c_{i+1} = 0.5x_i + 1.5c_i, \text{ so } c_{i+1} = 0.5(3c_i - 2c_{i-1}) + 1.5c_i = 3c_i - c_{i-1}$$

The recursive relation  $c_{i+1} = 3c_i - c_{i-1}$  tells us the solutions will all be integers. Some test-takers noticed that the values for  $c$ , which are 1, 3, 8, 21, 55, 144, 377, ..., are every other term from the Fibonacci sequence. The Fibonacci sequence, being the simplest nontrivial recursive sequence, appears a lot in recursive relations.

**3/2/10.** Nine cards can be numbered using positive half-integers ( $1/2, 1, 3/2, 2, 5/2, \dots$ ) so that the sum of the numbers on a randomly chosen pair of cards gives an integer from 2 to 12 with the same frequency of occurrence as rolling that sum on two standard dice. What are the numbers on the nine cards and how often does each number appear on the cards?

**Solution 1 by Megan Guichard (11/WA):** With nine cards, there are  ${}^9C_2 = 36$  possible ways to choose two cards. As it happens, there are also  $6 \cdot 6 = 36$  possible outcomes when two dice are rolled, with the following frequencies of occurrence:

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	2	3	4	5	6	5	4	3	2	1

Therefore, there must be exactly one pair of cards with a sum of 2, exactly two pairs of cards with a sum of 3, and so on.

Since all possible sums must be integers, the cards must be numbered either with all integers or with all odd integer multiples of  $1/2$ . Since half-integers are specifically mentioned, it seems a good assumption that all cards are numbered with odd integer multiples of  $1/2$  (at least until this

assumption is proven either true or false).

Operating under this assumption, there is only one way to get a sum of 2:  $1/2 + 3/2$ . Thus there must be exactly one card numbered  $1/2$  and exactly one numbered  $3/2$ . Then there must be two pairs of cards with sum 3; this can be accomplished by either  $1/2 + 5/2$  or  $3/2 + 3/2$ . However, we already know there is only one card numbered  $3/2$ , so there must be two ways to draw  $1/2 + 5/2$ . Only one card is numbered  $1/2$  so there must be two cards numbered  $5/2$ .

With the four cards we now have  $(1/2, 3/2, 5/2, 5/2)$  there are only two ways to draw a pair of cards with sum 4:  $3/2 + 5/2$  and  $3/2 + 5/2$  (there are two cards numbered  $5/2$ ). Therefore we need to add one more card that will create a sum of 4 when paired with an existing card. However the new card must be greater than  $5/2$ , because there are exactly the right number of cards less than or equal to  $5/2$  already; thus the new card must be  $7/2$ , which, when paired with  $1/2$ , yields the third way to get a sum of 4.

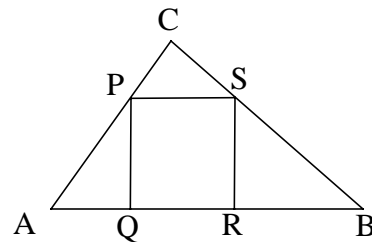
Continuing this line of reasoning shows that the remaining four cards must be numbered  $9/2, 9/2, 11/2,$  and  $13/2$ , meaning that the complete set of none cards is numbered as follows

$$(1/2, 3/2, 5/2, 5/2, 7/2, 9/2, 9/2, 11/2, 13/2)$$

With these cards, exactly one pair results in a sum of 2, exactly two pairings result in a sum of 3, three pairs yield a sum of 4, and so on, with each sum having the same frequency as on a pair of 6-sided dice.

**Editor's comments:** Erin Schram of the National Security Agency contributed this clever problem.

**4/2/10.** As shown on the figure, square  $PQRS$  is inscribed in right triangle  $ABC$ , whose right angle is at  $C$ , so that  $S$  and  $P$  are on sides  $BC$  and  $CA$ , respectively, while  $Q$  and  $R$  are on side  $AB$ . Prove that  $AB \geq 3QR$  and determine when equality occurs.



**Solution 1 by Suzanne Armstrong (11/MO):**

(i) Since  $\angle A \cong \angle CPS \cong \angle RSB$

$$\angle B \cong \angle CSP \cong \angle QPA$$

and  $\angle AQP \cong \angle C \cong \angle SRB = 90^\circ$

then  $\triangle ABC$  is similar to  $\triangle SRB$ .

(ii) Let  $\overline{AQ} = 1$  and let  $x = \overline{PQ} = \overline{PS} = \overline{QR} = \overline{RS}$ .

(iii) So

$$\frac{\overline{AQ}}{\overline{RS}} = \frac{\overline{PQ}}{\overline{BR}} \text{ and hence } \frac{1}{x} = \frac{x}{\overline{BR}} \text{ and therefore } \overline{BR} = x^2.$$

(iv)  $\overline{AB} = \overline{BR} + \overline{QR} + \overline{AQ} = x^2 + x + 1$



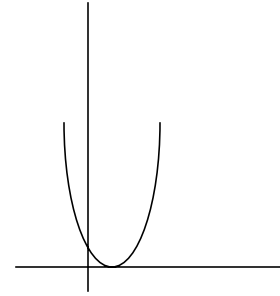
(v) Now to prove that  $\overline{AB} \geq 3\overline{QR}$ , we must show  $x^2 + x + 1 \geq 3x$ .

This is equivalent to  $x^2 - 2x + 1 \geq 0$  or  $(x - 1)^2 \geq 0$  which is clear and is further emphasized by its graph (at right).

(vi) As our last step, we determine when equality occurs. When

$\overline{QR} = 1 = \overline{AQ}$ , then  $\overline{AB} = 3\overline{QR} = 3$ ; also,

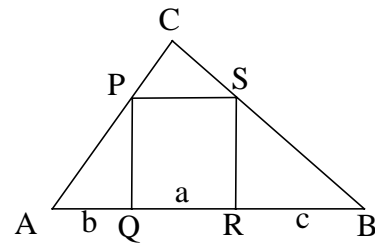
when  $\overline{AB} = 3\overline{QR} = 3$ , all of the triangles are isosceles  $45^\circ, 45^\circ, 90^\circ$ .



Thus,  $\overline{AB} \geq 3\overline{QR}$  with equality when  $\overline{AQ} = \overline{QR}$ .

**Solution 2 by Xuejing Chen (12/OK):** Observe that  $\angle APQ = \angle SBR$  and  $\angle PAQ = \angle BSC$ , so  $\triangle APQ$  is similar to  $\triangle SRB$ .

Thus  $\frac{a}{c} = \frac{b}{a}$  and  $a^2 = bc$ .



†Since  $(\sqrt{b} - \sqrt{c})^2 \geq 0$ , it follows that  $b + c \geq 2\sqrt{bc}$ .

So  $b + c \geq 2a$ , and  $b + c + a \geq 3a$ .

Thus  $\overline{AB} \geq 3\overline{QR}$ .

Now, if  $b = c$ , then  $b + c = 2a$  so we have equality. This occurs if  $\triangle CAB$  is isosceles.

**Solution 3 by Daniel Moraseski (11/FL):**

By similarity we have  $\frac{\overline{AQ}}{\overline{QR}} = \frac{\overline{AQ}}{\overline{PQ}} = \frac{\overline{AC}}{\overline{BC}}$  and  $\frac{\overline{RB}}{\overline{QR}} = \frac{\overline{RB}}{\overline{RS}} = \frac{\overline{BC}}{\overline{AC}}$

The Arithmetic Mean - Geometric Mean inequality states  $\frac{s+t}{2} \geq \sqrt{st}$  for nonnegative numbers  $s$  and  $t$ . [See † in Solution 2 above for proof.]

So  $\overline{AB} = \overline{QR} + \overline{AQ} + \overline{RB} = \overline{QR} \left( 1 + \frac{\overline{AC}}{\overline{BC}} + \frac{\overline{BC}}{\overline{AC}} \right) \geq \overline{QR} \left( 1 + 2\sqrt{\frac{\overline{AC}}{\overline{BC}} \cdot \frac{\overline{BC}}{\overline{AC}}} \right) = 3\overline{QR}$

Equality is satisfied when  $\frac{\overline{AC}}{\overline{BC}} = \frac{\overline{BC}}{\overline{AC}}$  because this is the equality condition for the AM-GM. This means  $\overline{AC} = \overline{BC}$  and it is an isosceles right triangle.

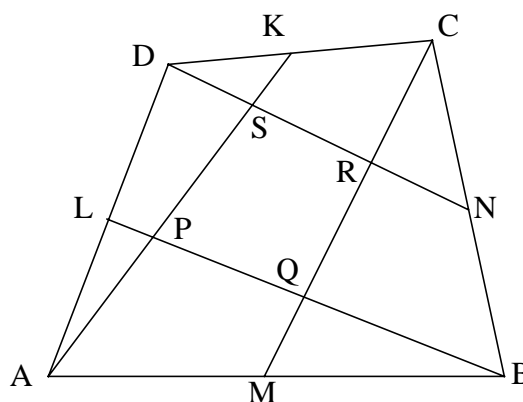
**Editor's Comments:** We are most grateful to Professor Hiroshi Okumura of the Maebashi Insti-

tute of Technology for this excellent problem. Professor Okumura is in charge of the Japanese counterpart of the USAMTS.

Many solutions used the inequality  $n + \frac{1}{n} \geq 2$  (for positive number  $n$ ) in one form or another.

Since  $(n - 1)^2 \geq 0$  we can write  $n^2 - 2n + 1 \geq 0$ , so  $n^2 + 1 \geq 2n$  and  $n + \frac{1}{n} \geq 2$ . If this fact was used in a solution, full credit for the solution required proof of this fact.

**5/2/10.** In the figure on the right,  $ABCD$  is a convex quadrilateral,  $K, L, M,$  and  $N$  are the midpoints of its sides, and  $PQRS$  is the quadrilateral formed by the intersections of  $AK, BL, CM,$  and  $DN$ . Determine the area of quadrilateral  $PQRS$  if the area of quadrilateral  $ABCD$  is 3000, and the areas of quadrilaterals  $AMQP$  and  $CKSR$  are 513 and 388, respectively.



**Solution 1 by Mary Tian (10/TX):**

Notation:  $[P_1P_2\dots P_v]$  denotes the area of polygon  $P_1P_2\dots P_v$ .

Connect  $AC$ . Because  $M, K$  are midpoints of  $AB, CD$  respectively, then  $[ACK] = [ACD]/2$ , and  $[CAM] = [CAB]/2$ .

Hence

$$[AMCK] = [ACK] + [CAM] = ([ACD]/2) + ([CAB]/2) = [ABCD]/2$$

Thus

$$\begin{aligned} [PQRS] &= [AMCK] - [AMQP] - [CKSR] \\ &= ([ABCD]/2) - [AMQP] - [CKSR] \\ &= (3000/2) - 513 - 388 \\ &= 599. \end{aligned}$$

So area of quadrilateral  $PQRS$  is 599.

**Editor's comment:** We are thankful to Professor Gregory Galperin of Eastern Illinois University for proposing an earlier version of this problem. His many contributions are most appreciated.