USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS

Round 3 - Year 14 - Academic Year 2002-2003

solutions edited by Dr. Erin Schram

1/3/14. The integer n, between 10000 and 99999, is $abcde$ when written in decimal notation. The digit a is the remainder when n is divided by 2, the digit b is the remainder when n is divided by 3, the digit c is the remainder when n is divided by 4, the digit d is the remainder when n is divided by 5, and the digit e is the remainder when n is divided by 6. Find n.

Comment: This problem is from a 1966 issue of *Abacus*, a Hungarian mathematics journal for middle school students.

Solution for 1/3/14 by Tamara Broderick (12/OH):

$n = abcde$.

$a = n \bmod 2$; therefore, a is equal to 0 or 1, but since $n \geq 10000$, a is 1, and n is odd.

Given this restriction, since $e = n \bmod 6$, it may be only $\{1, 3, 5\}$. When we divide e by 3, we get the same remainder as when we divide n by 3, so $b = e \bmod 3$. Also, because d is the remainder when n is divided by 5, d is determined entirely by e, the last digit of n.

e	1	3	5
b	1	0	2
d	1	3	0

The remainder, c, when n is divided by 4 is determined by the last two digits of n (since $100 \bmod 4 = 0$). So $c = de \bmod 4$.

de	11	33	05
c	3	1	1

Therefore, we have three possibilities for n:

a	1	1	1
b	1	0	2
c	3	1	1
d	1	3	0
e	1	3	5

The sum of the digits of n is congruent to n mod 3. Only in the first column of the above table is the sum of the digits, reduced mod 3, equal to b: $(1 + 1 + 3 + 1 + 1) \bmod 3 = 1$. So $n = 11311$.

2/3/14. Given positive integers *p*, *u*, and *v* such that $u^2 + 2v^2 = p$, determine, in terms of *u* and *v*, integers *m* and *n* such that $3m^2 - 2mn + 3n^2 = 24p$. (It is known that if p is any prime number congruent to 1 or 3 modulo 8, then we can find integers *u* and *v* such that $u^2 + 2v^2 = p$.)

Comments: This problem was developed by Dr. Robert Ward, a retired mathematician living in Maryland and active in the Ask Dr. Math program. It was inspired by Problem 2 for Grade 8 in the 1992 Spring Mathematical Competition of Bulgaria, which was called to our attention by the late Professor Ljubomir Davidov of Sofia.

Solution 1 for 2/3/14 by Subrahmanya Krishnamoorthy (10/NY):

We have $u^2 + 2v^2 = p$ and $3m^2 - 2mn + 3n^2 = 24p$. Dividing the latter equation throughout by three and multiplying the former equation throughout by eight, we have

$$m^2 - \frac{2}{3}mn + n^2 = 8p = 8u^2 + 16v^2.$$

Splitting the n^2 and rearranging gives

$$m^2 - \frac{2}{3}mn + \frac{1}{9}n^2 + \frac{8}{9}n^2 = 16v^2 + 8u^2.$$

This groups as

$$\left(m - \frac{n}{3}\right)^2 + 8\left(\frac{n}{3}\right)^2 = (4v)^2 + 8u^2.$$

Now we can see that the partitioning was wise, because setting the insides of the leftmost squares on each side equal and the insides of the rightmost squares on each side equal gives an easy way to make both sides equal:

$$m - \frac{n}{3} = 4v \quad \text{and} \quad \frac{n}{3} = u.$$

Thus, $n = 3u$ and $m = 4v + u$. Because both *u* and *v* are integers, *m* and *n* must also be integers by additive closure of integers.

Solution 2 for 2/3/14 by William Carlson (10/KS):

First, recognize that p, u, and v are integers that can be positive, and m and n are integers. It is also given that $u^2 + 2v^2 = p$ and $3m^2 - 2mn + 3n^2 = 24p$. So

$$m^2 - \frac{2}{3}mn + n^2 = 8p = 8u^2 + 16v^2.$$

To go about finding m and n in terms of u and v, I set up a table. For my p values I selected several primes congruent to 1 or 3 mod 8. I found u, v, m, and n values to go with those p values. At first, I didn't know which of those paired m and n values were m or n, because they are interchangeable in $m^2 - (2/3)mn + n^2 = 8p$, but they are not interchangeable when I was trying to find fuchtions $f(u, v)$ and $g(u, v)$ such that $m = f(u, v)$ and $n = g(u, v)$. The table given here has the m and n values in the proper order for my argument, though I had to switch the order of one pair in my original table.

p	3	11	17	19	41	43
u	1	3	3	1	3	5
v	1	1	2	3	4	3
m	3	9	9	3	9	15
n	5	7	11	13	19	17

From this table, it seemed that $m = 3u$.

So I put $3u$ in for m in the equation $m^2 - (2/3)mn + n^2 = 8u^2 + 16v^2$ to find out what n would be. This gave

$$\begin{aligned}(3u)^2 - (2/3)(3u)n + n^2 &= 8u^2 + 16v^2 \\ 9u^2 - 2un + n^2 &= 8u^2 + 16v^2 \\ u^2 - 2un + n^2 &= 16v^2 \\ (n-u)^2 &= (4v)^2 \\ n - u &= \pm 4v \\ n &= u \pm 4v\end{aligned}$$

Setting $n = u + 4v$ made all the m's and n's in the table work. Besides, if $m = 3u$ and $n = u + 4v$ are plugged back into the equation $m^2 - (2/3)mn + n^2 = 8u^2 + 16v^2$, the result is $8u^2 + 16v^2 = 8u^2 + 16v^2$.

One thing left: m and n have to be integers. If u and v are integers, and $m = 3u$ and $n = u + 4v$, then m and n must also be integers.

So $m = 3u$ and $n = u + 4v$.

Solution 3 for 2/3/14 by Alexander Yee (9/CA):

Given $3m^2 - 2mn + 3n^2 = 24p$ and $u^2 + 2v^2 = p$.

First, use the quadratic formula to solve $3m^2 - 2mn + 3n^2 = 24(u^2 + 2v^2)$ for *m* in terms of *n*, *u*, and *v*.

$$m = \frac{2n \pm \sqrt{(-2n)^2 - 4(3)(3n^2 - 24u^2 - 48v^2)}}{2(3)}$$

$$= \frac{n \pm 2\sqrt{-2n^2 + 18u^2 + 36v^2}}{3}$$

In order to get an integer, we must first eliminate all radicals. We need to select a value for *n* that when subsituted into the discriminant, $-2n^2 + 18u^2 + 36v^2$, will give a square of an integer. We can do that by setting *n* to $3u$.

$$m = \frac{(3u) \pm 2\sqrt{-2(3u)^2 + 18u^2 + 36v^2}}{3}$$

$$= \frac{3u \pm 2\sqrt{36v^2}}{3}$$

$$= \frac{3u \pm 12v}{3}$$

$$= u \pm 4v$$

We have found two solutions. There are eight in all. Setting *n* to $-3u$ also eliminates the radical and gives $m = -u \pm 4v$. Notice that the equation $3m^2 - 2mn + 3n^2 = 24p$ is symmetrical in *m* and *n*. Therefore, we can switch the values of *m* and *n* and still have a solution. That gives eight solutions for (m, n): $(u + 4v, 3u)$, $(u - 4v, 3u)$, $(-u + 4v, -3u)$, $(-u - 4v, -3u)$, $(3u, u + 4v)$, $(3u, u - 4v)$, $(-3u, -u + 4v)$, and $(-3u, -u - 4v)$.

Solution 4 for 2/3/14 by Mircea (Bobby) Georgescu (9/CA):

Notation: $x|y$ means that *x* is a factor of *y*.

We are given $3m^2 - 2mn + 3n^2 = 24p$ (equation 1). So $2mn = 3m^2 + 3n^2 - 24p = 3(m^2 + n^2 - 8p)$. Therefore, $3|2mn$. Since 3 is prime and 3 is not a factor of 2, $3|m$ and/or $3|n$.

Let us consider the case where $3|m$ (the $3|n$ case has a similar solution). We set $m = 3k$ for some integer *k*. Substituting $m = 3k$ in equation 1 gives $27k^2 - 6kn + 3n^2 = 24p$, which simplifies to $9k^2 - 2kn + n^2 = 8p$ (equation 2).

Equation 2 is equivalent to $k^2 - 2kn + n^2 = 8(p - k^2)$, so $8|(k^2 - 2kn + n^2)$. This is $8|(k - n)^2$, which means that $4|(k - n)$. So $k - n = 4j$ for some integer *j*.

We substitute $n = k - 4j$ and equation 2 becomes $8k^2 + 16j^2 = 8p$, which is equivalent to $k^2 + 2j^2 = p$ (equation 3). We know that $u^2 + 2v^2 = p$, so setting $k = u$ and $j = v$ gives a solution to equation 3. This gives $m = 3u$ and $n = u - 4v$ as a solution to equation 1.

3/3/14. Determine, with proof, the rational number $\frac{m}{n}$ that equals

$$\frac{1}{1\sqrt{2}+2\sqrt{1}}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\ldots+\frac{1}{4012008\sqrt{4012009}+4012009\sqrt{4012008}}.$$

Comment: This problem was inspired by a similar problem in *Matlap*, Transylvania's outstanding Hungarian-language mathematics journal for middle school and high school students.

Solution 1 for 3/3/14 by Eric Stansifer (10/FL):

All of the terms of the expression given were of the form

$$\frac{1}{a\sqrt{a+1}+(a+1)\sqrt{a}}.$$

I decided to simplify this into an easier-to-manipulate form, so I multiplied the numerator and denominator by the conjugate of the denominator:

$$\begin{aligned}
\frac{1}{a\sqrt{a+1}+(a+1)\sqrt{a}} &= \frac{1}{a\sqrt{a+1}+(a+1)\sqrt{a}}\times\frac{a\sqrt{a+1}-(a+1)\sqrt{a}}{a\sqrt{a+1}-(a+1)\sqrt{a}}\\
&= \frac{a\sqrt{a+1}-(a+1)\sqrt{a}}{a^2(a+1)+(a+1)^2(a)}\\
&= \frac{a\sqrt{a+1}}{a^2(a+1)+(a+1)^2(a)}-\frac{(a+1)\sqrt{a}}{a^2(a+1)+(a+1)^2(a)}\\
&= \frac{\sqrt{a+1}}{a(a+1)-(a+1)^2}-\frac{\sqrt{a}}{a^2-a(a+1)}\\
&= \frac{\sqrt{a+1}}{-(a+1)}-\frac{\sqrt{a}}{-a}\\
&= \frac{1}{\sqrt{a}}-\frac{1}{\sqrt{a+1}}
\end{aligned}$$

When each term of the given expression is modified in the above fashion, it yields a telescoping series which simplifies to a rational form. The modified series is

$$\left(\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right)+\ldots+\left(\frac{1}{\sqrt{4012008}}-\frac{1}{\sqrt{4012009}}\right).$$

Canceling out all but the first and last terms yields

$$\frac{1}{\sqrt{1}}-\frac{1}{\sqrt{4012009}} = \frac{1}{1}-\frac{1}{2003} = \frac{2002}{2003}.$$

The answer is 2002/2003.

Solution 2 for 3/3/14 by Hyun-Soo Kim (10/NJ):

The sum given in the problem can be written in summation notation as

$$\sum_{n=1}^{k} \frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}$$

where in this case $k = 4012008$.

Upon calculating the sum for smaller values of *k*, such as 2 and 3, a pattern was established. The pattern is that

$$\sum_{n=1}^{k}\left(\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right) = 1-\frac{1}{\sqrt{k+1}}.$$

We will prove by induction that the above conjecture is true.

Base case:

$k = 1$.

$$\sum_{n=1}^{1}\left(\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right) = \frac{1}{1\sqrt{2}+2\sqrt{1}} = \frac{1}{2+\sqrt{2}} = \frac{2-\sqrt{2}}{2} = 1-\frac{1}{\sqrt{2}} = 1-\frac{1}{\sqrt{k+1}}.$$

Assuming that the induction hypothesis holds for a given value *k*, prove that it holds for the value $k+1$:

$$\sum_{n=1}^{k+1}\left(\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right) = \sum_{n=1}^{k}\left(\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right)+\frac{1}{(k+1)\sqrt{k+2}+(k+2)\sqrt{k+1}}$$

$$= 1-\frac{1}{\sqrt{k+1}}+\frac{1}{(k+1)\sqrt{k+2}+(k+2)\sqrt{k+1}}$$

$$= 1+\frac{(-1)((k+1)\sqrt{k+2}+(k+2)\sqrt{k+1})+\sqrt{k+1}}{(\sqrt{k+1})((k+1)\sqrt{k+2}+(k+2)\sqrt{k+1})}$$

$$= 1+\frac{-(k+1)\sqrt{k+1}-(k+1)\sqrt{k+2}}{(k+1)\sqrt{k+1}\sqrt{k+2}+(k+1)(k+2)}$$

$$= 1+\frac{(-1)(k+1)(\sqrt{k+1}+\sqrt{k+2})}{(\sqrt{k+2})(k+1)(\sqrt{k+1}+\sqrt{k+2})}$$

$$= 1-\frac{1}{\sqrt{k+2}}$$

$$= 1-\frac{1}{\sqrt{(k+1)+1}}$$

So by induction, our conjecture is true. The rational number that equals the given sum is:

$$\sum_{n=1}^{4012008}\left(\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right) = 1-\frac{1}{\sqrt{4012008+1}} = 1-\frac{1}{2003} = \frac{2002}{2003}.$$

4/3/14. The vertices of a cube have coordinates $(0, 0, 0)$, $(0, 0, 4)$, $(0, 4, 0)$, $(0, 4, 4)$, $(4, 0, 0)$, $(4, 0, 4)$, $(4, 4, 0)$, and $(4, 4, 4)$. A plane cuts the edges of this cube at the points $(0, 2, 0)$, $(1, 0, 0)$, $(1, 4, 4)$, and two other points. Find the coordinates of the other two points.

Comments: This problem was contributed by Prof. George Berzsenyi, the founder and problem editor of the USAMTS. We are thankful for all the ways in which Prof. Berzsenyi supports the USAMTS.

Solution 1 for 4/3/14 by Kevin Lin (10/OH):

The general equation of the plane is $aX + bY + cZ = d$.

Put in the three points on the plane: $(0, 2, 0)$, $(1, 0, 0)$, and $(1, 4, 4)$:

$$(a)(0) + (b)(2) + (c)(0) = d$$
$$(a)(1) + (b)(0) + (c)(0) = d$$
$$(a)(1) + (b)(4) + (c)(4) = d$$

Solve for *a*, *b*, and *c* in terms of *d*. We have

$$a = d, \quad b = \frac{d}{2}, \quad c = -\frac{d}{2}.$$

Setting $d = 2$, the equation of the plane is

$$2X + Y - Z = 2.$$

The cube has 12 edges, each defined by a pair of constraints:

Edge 1 $X = 0$ & $Y = 0$

Edge 2 $X = 0$ & $Y = 4$

Edge 3 $X = 0$ & $Z = 0$ The plane cuts this edge at known point $(0, 2, 0)$

Edge 4 $X = 0$ & $Z = 4$

Edge 5 $X = 4$ & $Y = 0$

Edge 6 $X = 4$ & $Y = 4$

Edge 7 $X = 4$ & $Z = 0$

Edge 8 $X = 4$ & $Z = 4$

Edge 9 $Y = 0$ & $Z = 0$ The plane cuts this edge at known point $(1, 0, 0)$

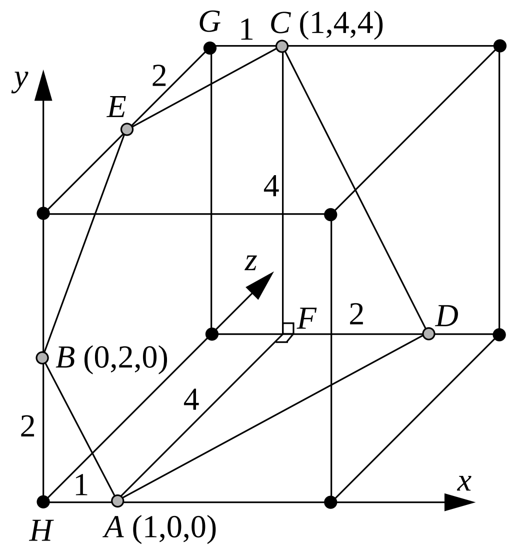
Edge 10 $Y = 0$ & $Z = 4$

Edge 11 $Y = 4$ & $Z = 0$ The plane cuts this edge at known point $(1, 4, 4)$

Edge 12 $Y = 4$ & $Z = 4$

If an edge of the cube is cut by the plane, every coordinate of the intersection point has to be between 0 and 4, inclusive, or that point is not on the cube. Checking the intersection points of the nine edges without known points, we find that the plane cuts edge 2 at $(0, 4, 2)$ and cuts edge 10 at $(3, 0, 4)$.

Solution 2 for 4/3/14 by Karl Jiang (12/FL):



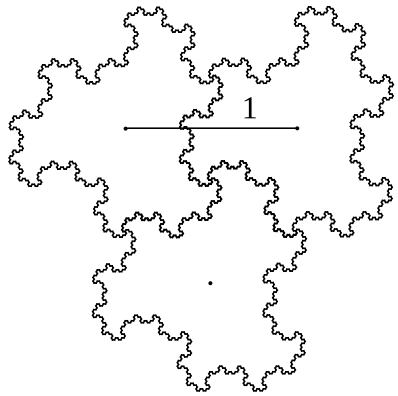
To the left is a figure of the cube suggested. The three known intersection points of the plane and the edges are *A*, *B*, and *C*, shown labeled with their coordinates. If two intersection points share a face of the cube, the line segment drawn between them is the intersection of the plane and that face. We can see that for faces opposite each other on the cube, the slopes of those plane-face intersection segments are going to be the same. In this way, we can use the proportions of those slopes to get the remaining two plane-edge intersection segments.

Line segment $\overline{AB}$ is the intersection of the plane and a face. Naming the vertex with coordinates $(0, 0, 0)$ as *H*, we see that $BH = 2$ and $AH = 1$. The opposite face must contain a right triangle *CDF* similar to triangle *BAH*, with point *F* directly below point *C* at coordinates $(1, 0, 0)$ and with line segment $\overline{CD}$ being the intersection of the plane and the face. $CF = 4$ and $DF/CF = AH/BH = 1/2$, so $DF = 2$ and we can conclude that $D = (3, 0, 4)$.

Line segment $\overline{AD}$ is the intersection of the plane and a face. We see that $AF = 4$ and $DF = 2$. The opposite face must contain a right triangle *CEG* similar to triangle *DAF*, with point *G* at the vertex $(0, 4, 4)$ and with line segment $\overline{CE}$ being the intersection of the plane and the face. $CG = 1$ and $EG/CG = AF/DF = 2$, so $EG = 2$ and we can conclude that $E = (0, 4, 2)$.

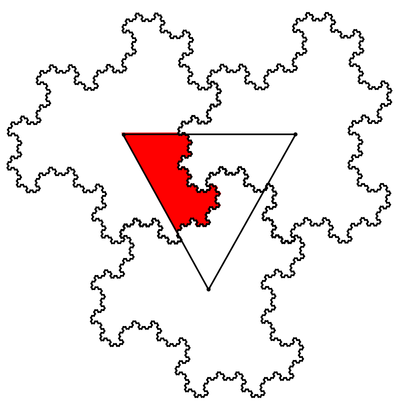
Answer: The two other intersection points are $(3, 0, 4)$ and $(0, 4, 2)$.

5/3/14. A fudgeflake is a planar fractal figure with 120˚ rotational symmetry such that three identical fudgeflakes in the same orientation fit together without gaps to form a larger fudgeflake with its orientation 30˚ clockwise of the smaller fudgeflakes' orientation, as shown on the right. If the distance between the centers of the original three fudgeflakes is 1, what is the area of one of those three fudgeflakes? Justify your answer.



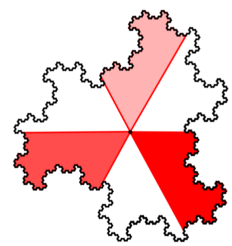
Comments: This problem was written by Dr. Erin Schram, the solutions editor of the USAMTS. The fudgeflake came from **The Fractal Geometry of Nature** by Benoit Mandelbrot, who credits a 1970 paper by Davis and Knuth in the *Journal of Recreational Mathematics*.

Solution 1 for 5/3/14 by Guy David (9/FL):

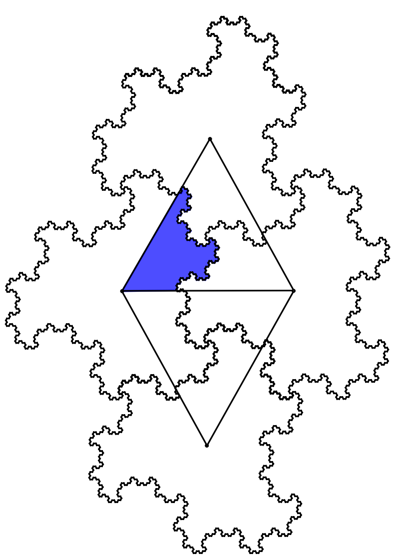


We can first connect the centers of the three fudgeflakes, creating an equilateral triangle with side length 1. We note the region that the triangle cuts out from one fudgeflake, which is shaded in red in the diagram to the left.

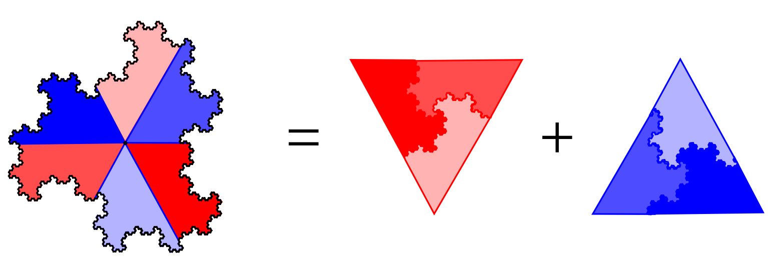
As the fudgeflakes have 120˚ rotational symmetry, three copies of this region occur in the fudgeflake.



We can put a fourth fudgeflake above the others. It fits the one to its lower left just like the other two fudgeflakes fit together, so their centers are 1 apart. Likewise the centers of that fudgeflake and the fudgeflake to its lower right are 1 apart. This gives us a different equilateral triangle that cuts out a different region, shaded in blue.



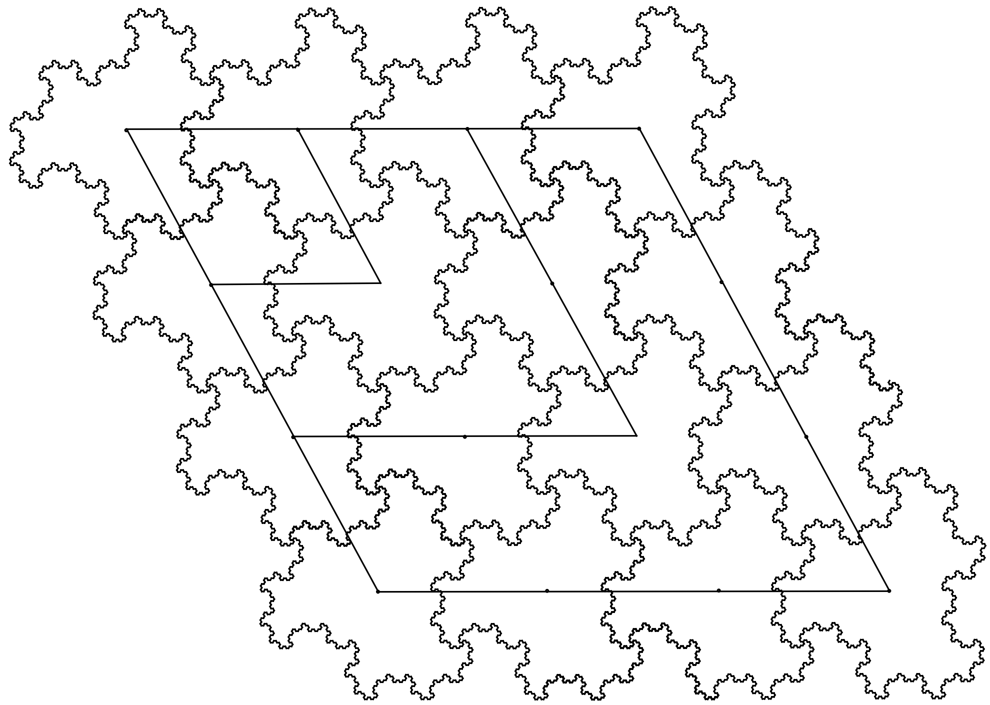
This blue region can be found in three places within the fudgeflake. Three copies of the red region and three copies of the blue region combine to form a full fudgeflake. The sum of the areas of all three red pieces is equal to the area of the equilateral triangle with side 1, and the sum of the areas of all three blue pieces is the same. Therefore, the area of one fudgeflake is twice the area of an equilateral triangle with side 1: $2 \times \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$.



Solution 2 for 5/3/14 by Boris Hanin (10/ID):

Observe that we can completely tesselate any part of the plane using the fractal. Also, by connecting the centers of adjacent fractals, we get a grid made of equilateral triangles with side 1.

Consider the sequence of nested rhombuses shown below.



An $n \times n$ rhombus is contained in $(n+1)^2$ fractals and contains $(n-1)^2$ fractals. This means that n^2 fractals arranged in a rhombus pattern contain an $(n-1) \times (n-1)$ rhombus and are contained in an $(n+1) \times (n+1)$ rhombus. If we take those two rhombuses, their areas give us an upper bound and a lower bound for the area of the n^2 fractals.

The area of a rhombus with side a is $a^2 \sin\alpha$ where α is the angle between two adjacent sides. Observing that the grid of equilateral triangles between the centers of the fractals gives $\alpha = 60°$, the area of a rhombus with side a is $\frac{a^2\sqrt{3}}{2}$. Denoting the area of a fractal as A, we get

$$\frac{(n-1)^2\sqrt{3}}{2} < n^2 A < \frac{(n+1)^2\sqrt{3}}{2}.$$

Equivalently,

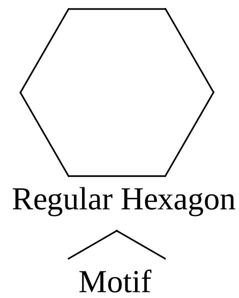
$$\frac{(n-1)^2\sqrt{3}}{2n^2} < A < \frac{(n+1)^2\sqrt{3}}{2n^2}.$$

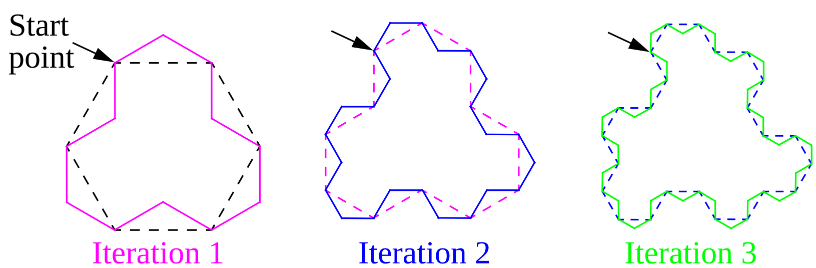
Observe that $\lim_{n\to\infty} \frac{(n-1)^2\sqrt{3}}{2n^2} = \lim_{n\to\infty} \frac{(n+1)^2\sqrt{3}}{2n^2} = \frac{\sqrt{3}}{2}$, so letting $n \to \infty$ and applying the squeeze theorem, we get $A = \frac{\sqrt{3}}{2}$.

Comment from the solutions editor: One rare type of solution, such as the one below, requires finding a reference that describes the construction of a fudgeflake, such as Mandelbrot's **The Fractal Geometry of Nature**. Outside references like this are perfectly acceptable, though a solution should always cite any special references used.

Solution 3 for 5/3/14 by Jason Allen (12/MN):

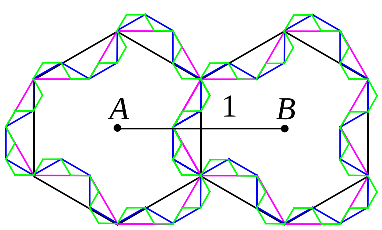
The fudgeflake is constructed as follows. You begin with a regular hexagon like the one pictured at right. Then for each of the six sides, you replace the side with the motif at right, alternating directions so that the motif bends in at one side, out at the adjacent side, in again for the third side, out again for the fourth side, and so on. The motif has a 120˚ angle and its missing base is as long as a side of the hexagon. Repeat this again and again with the resulting figure, scaling the motif smaller at each iteration, and always starting with the same vertex and with the same direction at each iteration. Three iterations are shown below.

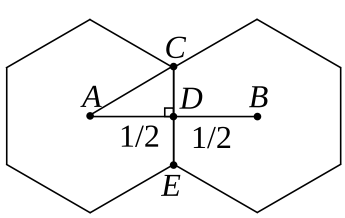




In every iteration, we are adding a certain number of motif triangles to the figure and we are subtracting the same number of triangles with the same area from the figure, so the area in each successive iteration remains constant. Note that the center does not move between iterations.

If we transform two identical hexagons that share a side into fudgeflakes, if we coordinate their motif patterns, the resulting fudgeflakes fit together the same way that the fudgeflakes in this problem fit together. Since the fudgeflakes have the same area as the initial hexagons, we just need to find the area of those hexagons.





Label the centers of the fudgeflakes as *A* and *B*. We are given that $AB = 1$ and we deduced that *A* and *B* are the centers of congruent regular hexagons that share a side and have the same area as the fudgeflakes. Congruent hexagons have congruent apothems, so $AD = BD = 1/2$ and line segment $\overline{AD}$ is per-perpendicular to side $\overline{CE}$. Triangle *ACE* is equilateral, since the hexagon is regular, so $m\angle ACD = 60°$. Triangle *ACD* is a 30-60-90 triangle, so $CD = \sqrt{3}/6$. The area of triangle *ACD* is $(1/2) \times (\sqrt{3}/6) \times (1/2) = \sqrt{3}/24$. The hexagon centered at point *A* is made up of 12 triangles congruent to triangle *ACD*, so the area of the hexagon is $\sqrt{3}/2$. Therefore, the area of one of the fudgeflakes is $\dfrac{\sqrt{3}}{2}$.