USA Mathematical Talent Search
CREDITS and QUICK ANSWERS
Round 1 — Year 15 — Academic Year 2003–2004

1/1/15. Let $f(x) = \lfloor x \rfloor + \lfloor 10x \rfloor + \lfloor 100x \rfloor + \lfloor 1000x \rfloor$ for all real numbers x, where for any real number r, $\lfloor r \rfloor$ means the greatest integer less than or equal to r. For example, $f(\pi) = 3+31+314+3141 = 3489$. Find, with proof, a positive integer n less than 2003 such that $f(x) = n$ has no solution for x, but $f(y) = n + 11$ and $f(z) = n + 111$ have solutions for y and z.

The function $f(x) = \lfloor x \rfloor + \lfloor 10x \rfloor + \lfloor 100x \rfloor + \lfloor 1000x \rfloor$ was invented by George Berzsenyi, the founder of the USAMTS.

Since f is constant over all real numbers in the half-open interval $[i/1000, (i+1)/1000)$ for all integers i, to find all possible values of f we need look only at the values of $f(i/1000)$ for integers i. The values of $f(i/1000)$ and $f((i+1)/1000)$ are consecutive integers unless $i+1$ is divisible by 10. $n = 1108$ is in the gap between $f(999/1000) = 1107$ and $f(1000/1000) = 1111$, $n+11 = 1119 = f(1008/1000)$, and $n+111 = 1219 = f(1099/1000)$.

2/1/15. Find all primes p for which $6p+1$ is the fifth power of an integer. Prove that you found all of them.

This problem was also devised by George Berzsenyi.

Suppose $6p+1$ is a fifth power n^5. This gives $6p = n^5 - 1 = (n-1)(n^4+n^3+n^2+n+1)$. The second factor, $n^4+n^3+n^2+n+1$, is greater than 6, since n is at least 2. So p must be a factor of $n^4+n^3+n^2+n+1$. This leaves $n-1$ as a factor of 6, so n could be only 2, 3, 4, or 7. Of those, only $n = 7$ works, giving $7^5 = 6 \cdot 2801 + 1$. Thus, the only answer for p is the prime 2801.

3/1/15. We attempted to arrange the integers 1, 2, 3, ..., 12 around a circle so that all sums of pairs of adjacent integers are either

(a) all perfect squares, or

(b) all triangular numbers, which are numbers of the form $n(n+1)/2$.

One case out of (a) and (b) succeeded. Which one? For the successful case, show a valid arrangement. For the unsuccessful case, explain why it is impossible.

This problem was also devised by George Berzsenyi.

Case (b) succeeded.

(a) The only summation involving 12 that sums to a perfect square is $4 + 12 = 16$. Since we would need two sums involving 12 in order to arrange the integers 1, 2, 3, ..., 12 around a circle, this case is impossible.

(b) The triangular numbers are 1, 3, 6, 10, 15, 21, 28, etc. The circle to the right has the numbers 1 through 12 arranged so that all sums of pairs of adjacent numbers are triangular.

6 9
4 12
11 3
10 7
5 8
1 2

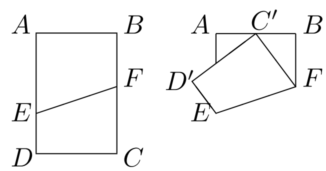
4/1/15. Two players play a game. Each player, in turn, has to name a positive integer that is less than the previous number but at least half the previous number. The player who names the number 1 loses. If the first player starts by naming 2003 and after that both players play with the the best strategy, who wins? Describe the strategy and prove it works.

This problem was invented by David Grabiner, a mathematician at NSA.

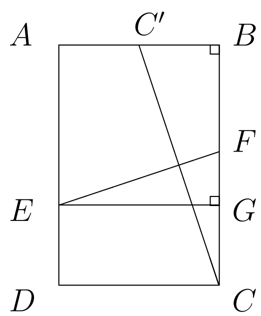
The second player will win.

The best strategy is to name only numbers of the form $3(2^n) - 1$, which are 1535, 767, 383, 191, 95, 47, 23, 11, 5, and 2. If a player names a number of the form $3(2^n) - 1$, then the other player has to name a number between $3(2^{n-1})$ and $3(2^n) - 2$, inclusive. No matter which of those numbers the other player names, the strategic player can legally respond by naming $3(2^{n-1}) - 1$. The last number the strategic player names will be $3(2^0) - 1 = 2$. The other player can respond only by naming 1, so that player will lose. In the game from the problem, the first player started with the nonstrategic number 2003, so the second player gets to claim the best strategy by naming 1535.

5/1/15. A rectangular piece of paper, $ABCD$, is folded along line segment $\overline{EF}$, where point E is on side $\overline{AD}$ and point F is on side $\overline{BC}$, so that C ends up at the midpoint of side $\overline{AB}$. Determine the length of $\overline{EF}$ if the length of $\overline{AB}$ is 240 and the length of $\overline{BC}$ is 288.



This problem is based on Problem 2.3.3 of *Traditional Japanese Problems of the 18th and 19th Century* by Fukagawa and Rigby.



A fold reflects points; therefore, the line $\overleftrightarrow{EF}$ is the perpendicular bisector of line segment $\overline{CC'}$. The length of $\overline{CC'}$ is $\sqrt{(120^2 + 288^2)} = 312$. Angle $\angle CC'B$ is congruent to angle $\angle EFC$, so there exists a point G on ray $\overrightarrow{FC}$ such that triangle EFG is a right triangle similar to triangle $CC'B$. Side $\overline{EG}$ is parallel to side $\overline{AB}$, so $EG = 240$. By ratios of corresponding lengths on similar triangles, $EF/EG = CC'/BC$. Therefore, $EF = (240)(312)/288 = 260$.