USA Mathematical Talent Search
CREDITS and QUICK ANSWERS
Round 2 — Year 15 — Academic Year 2003–2004

1/2/15. The faces of 27 unit cubes are painted red, white, and blue in such a manner that we can assemble them into three different configurations: a red $3 \times 3 \times 3$ cube, a white $3 \times 3 \times 3$ cube, and a blue $3 \times 3 \times 3$ cube. Determine, with proof, the number of unit cubes on whose faces all three colors appear.

This problem was devised by George Berzsenyi, Professor Emeritus of Rose-Hulman Institute for Technology and founder of the USAMTS. We are thankful for all the work Prof. Berzsenyi puts into this contest.

Painting the surface of three $3 \times 3 \times 3$ cubes requires 162 square units, which equals the total surface area of the 27 unit cubes. So no face is left unpainted. In the white $3 \times 3 \times 3$ cube the eight cubes on the corner each have three white faces, the twelve cubes on the edges each have two white faces, the six cubes on the face centers each have one white face, and the one cube in the center has zero white faces. The same happens for red and blue. The only way for a unit cube to miss a color is if it serves as the center cube of that color's $3 \times 3 \times 3$ cube. A cube missing a color must contain three faces of each of the other two colors, so each center of a monotone $3 \times 3 \times 3$ cube is distinct. Thus, exactly three cubes have two colors and exactly twenty-four cubes have three colors.

2/2/15. For any positive integer n, let $s(n)$ denote the sum of the digits of n in base 10. Find, with proof, the largest n for which $n = 7s(n)$.

This problem was inspired by a "Problem of the Month" proposal of Professor Béla Bajnok of Gettysburg College.

Consider a positive integer n with at most d digits. We write it as $a_{d-1}a_{d-2}\ldots a_2a_1a_0$ with each a_i being a digit from 0 to 9, starting with leading zeros if n has fewer than d digits. Then $n = \sum_{i=0}^{d-1} 10^i a_i$ and $s(n) = \sum_{i=0}^{d-1} a_i$. The equation $n = 7s(n)$ becomes

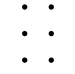
$$10^{d-1}a_{d-1} + \ldots + 100a_2 + 10a_1 + a_0 = 7a_{d-1} + \ldots 7a_2 + 7a_1 + 7a_0. \tag{1}$$

Cancelling like terms on opposite sides leaves

$$(10^{d-1} - 7)a_{d-1} + \ldots + 93a_2 + 3a_1 = 6a_0. \tag{2}$$

Since the maximum value of the righthand side of equation (2) is $6 \cdot 9 = 54$, we have that $a_i = 0$ for all $i > 1$. This leaves $3a_1 = 6a_0$, that is $a_1 = 2a_0$. The largest digits that work for a_1 and a_0 are $a_1 = 8$ and $a_0 = 4$, giving $n = 84$.

3/2/15. How many circles in the plane contain at least three of the nine points $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$, $(2, 2)$? Rigorously verify that no circle was skipped or counted more than once in the result.

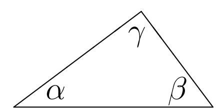


We are thankful to Professor Harold B. Reiter, the President of Mu Alpha Theta, for this problem.

There are 34 circles: one through $(0,0)$, $(0,2)$, $(2,0)$, and $(2,2)$; one through $(0,1)$, $(1,0)$, $(1,2)$, and $(2,1)$; four like the one through $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$; four like the one through $(0,0)$, $(0,1)$, $(2,0)$, and $(2,1)$; four like the one through $(0,0)$, $(0,1)$, $(1,2)$, and $(2,2)$; eight like the one through $(0,0)$, $(1,1)$, and $(1,2)$; four like the one through $(0,0)$, $(0,2)$, and $(1,1)$; four like the one through $(0,0)$, $(0,2)$, and $(1,2)$; and four like the one through $(0,0)$, $(1,2)$, and $(2,1)$.

To verify we did not skip a circle, we count that every set of three points has been covered. There are $\binom{9}{3} = 84$ such sets. The 14 circles with four points each account for $4 \cdot 14 = 56$ sets. The 20 circles with three points each account for 20 sets. There are eight lines that cover three points each, which do not count as circles. $56+20+8 = 84$, accounting for every set of three points.

4/2/15. In how many ways can one chose three angle sizes, α, β, and γ, with $\alpha \le \beta \le \gamma$ from the set of integral degrees, $1°$, $2°$, $3°$, …, $178°$, such that those angle sizes correspond to the angles of a nondegenerate triangle? How many of the resulting triangles are acute, right, and obtuse, respectively?



The first question came from Hungary's famous high school mathematics journal *KöMaL* more than a century ago. The second question was added by George Berzsenyi.

For acute triangles, angle γ ranges from $60°$ to $89°$, angle β ranges from $\lceil (180° - \gamma)/2 \rceil$ to γ, and angle α is $180° - \gamma - \beta$. That adds up as $1+2+4+5+7+8+\cdots+43+44 = 675$ acute triangles. For right triangles, angle γ is $90°$, angle β ranges from $45°$ to $89°$, and angle α is $90° - \beta$. That gives 45 right triangles. For obtuse triangles, angle γ ranges from $91°$ to $178°$, angle β ranges from $\lceil (180° - \gamma)/2 \rceil$ to $179° - \gamma$, and angle α is $180° - \gamma - \beta$. That adds up as $44+44+43+43+\ldots+1+1 = 1980$ obtuse triangles. That gives 2700 triangles total.

5/2/15. Clearly draw or describe a convex polyhedron that has exactly three pentagons among its faces and the fewest edges possible. Prove that the number of edges is a minimum.

This problem was inspired by a problem from the keynote address given by Dr. Robert Geretschläger at the Congress of the World Federation of National Mathematics Competitions in Melbourne, Australia, in August 2002. Dr. Geretschlänger is the Leader of Austria's IMO team. His permission to use the original problem is most appreciated.

Each pentagon has five edges on it, but any pair of pentagons can share one edge. That gives at least 12 edges on the three pentagons. At least one vertex on each pentagon does not touch another pentagon. Such a vertex has at least one more edge from it, which we will count as half an edge, because we might also count the other end of that edge elsewhere. Thirteen and a half edges round up to a minimun of 14 edges.

There are two topologically-distinct convex polyhedra with three pentagonal faces and 14 edges. The faces of both polyhedra consist of three pentagons, one quadrilateral, and three triangles. We can construct either out of a triangular prism, one by making two triangular cuts on adjacent corners of a triangle and the other by making two triangular cuts on a single corner, as shown on the right.