

**USA Mathematical Talent Search
PROBLEMS**

Round 4 — Year 15 — Academic Year 2003–2004

1/4/15. Find, with proof, the smallest positive integer n for which the sum of the digits of $29n$ is as small as possible.

2/4/15. For four integer values of n greater than six, there exist right triangles whose side lengths are integers equivalent to 4, 5, and 6 modulo n , in some order. Find those values. Prove that at most four such values exist. Also, for at least one of those values of n , provide an example of such a triangle.

3/4/15. Find a nonzero polynomial $f(w, x, y, z)$ in the four indeterminates $w, x, y,$ and z of minimum degree such that switching any two indeterminates in the polynomial gives the same polynomial except that its sign is reversed. For example, $f(z, x, y, w) = -f(w, x, y, z)$. Prove that the degree of the polynomial is as small as possible.

4/4/15. For each nonnegative integer n define the function $f_n(x)$ by

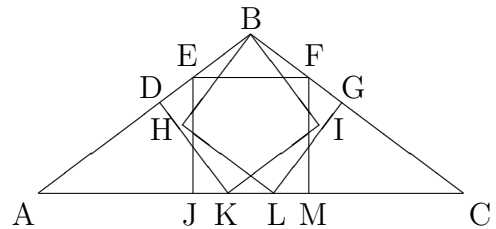
$$f_n(x) = \sin^n(x) + \sin^n\left(x + \frac{2\pi}{3}\right) + \sin^n\left(x + \frac{4\pi}{3}\right)$$

for all real numbers x , where the sine functions use radians. The functions $f_n(x)$ can be also expressed as polynomials in $\sin(3x)$ with rational coefficients. For example,

$$\begin{aligned} f_0(x) &= 3, & f_1(x) &= 0, & f_2(x) &= \frac{3}{2}, & f_3(x) &= -\frac{3}{4}\sin(3x), \\ f_4(x) &= \frac{9}{8}, & f_5(x) &= -\frac{15}{16}\sin(3x), & f_6(x) &= \frac{27}{32} + \frac{3}{16}\sin^2(3x), \end{aligned}$$

for all real numbers x . Find an expression for $f_7(x)$ as a polynomial in $\sin(3x)$ with rational coefficients, and prove that it holds for all real numbers x .

5/4/15. Triangle ABC is an obtuse isosceles triangle with the property that three squares of equal size can be inscribed in it as shown on the right. The ratio AC/AB is an irrational number that is the root of a cubic polynomial. Determine that polynomial.



Complete, well-written solutions to at least two of the problems above, accompanied by a Cover Sheet, should be mailed to:

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and postmarked by Sunday, 14 March 2004. Each participant is expected to develop solutions without help from others. For the cover sheet and other details, see the USAMTS web site: <http://www.nsa.gov/programs/mepp/usamts.html>.