USA Mathematical Talent Search

Round 1 Grading Rubric

Year 33 — Academic Year 2021–2022

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/1/33:

Award **5 points** for the correct configuration of numbers. No justification is required. Withhold **1 point** for each incorrect entry.

If a student has many errors, but they have both the correct starting and ending points of the path (possibly reversed), award a total of **1 point**.

Problem 2/1/33:

Note: As defined in the official solution, a good triple is a set of distinct integers a, b, c between 1 and n, inclusive, all of the same color, such that $2a + b = c$.

2 points: Student provides a coloring for $n = 14$ in which there are no good triples (**1 point**) and provides supporting explanation as needed regarding how they came up with the coloring and why there are no good triples (**1 point**). If the student doesn't explain how they came up with their coloring for $n = 14$ or why it doesn't have any good triples, don't automatically deduct any points, but consider a deduction if the solution is otherwise borderline.

3 points: Student proves that if $n = 15$, then there must be a good triple. Award (at least) **1 point** of partial credit if the student shows how the color choices for some numbers affect the possible colors for other numbers. Whether you award **1 point** or **2 points** of partial credit will largely depend on how close the student is to a complete and correct solution.

Note: If the student finds the coloring for $n = 14$ that doesn't contain any good triples, and then shows that coloring 15 either red or blue would lead to a good triple, award a total score of **3 points** unless the student proves that the coloring for $n = 14$ is the only such coloring that doesn't have any good triples, in which case a score of **5 points** is appropriate.

Note: If the student obtains the correct answer, but doesn't provide any explanation, award **1 point**.

Problem 3/1/33:

3 points: Student shows that the set of squarefree integers (or some equivalent set) maximizes the number of elements of S. Award **1 point** for stating this result, and **2 points** for the justification. Award **1 point** out of the **2 points** for the justification if the student has significant constructive progress towards the complete justification. An example of significant constructive progress would be recognizing that if $f(n)$ is the squarefree part of n (i.e., $f(n) = \frac{n}{d}$, where d is the greatest divisor of n that is a perfect square), then mn is a perfect square if and only if $f(m) = f(n)$.

Note: It is not sufficient to say that if an^2 is in S, then a cannot be in S. For full credit, the student needs to explain why it is impossible to include both am^2 and an^2.

2 points: Student correctly counts the number of elements in S. Award **1 point** of partial credit if the counting (**with justification**) is almost correct, but there is a minor flaw. In other words, if a student doesn't explain their counting strategy and is off by 1, don't give the partial credit.

Note: The counting method in the official solution was reasonably common, but students used a variety of counting methods, including creating a computer program, which is valid. Award full credit if the student describes the algorithm but does not include the code for the program. In the future, students should include their code.

Note: If the student obtains the correct answer, but doesn't provide any explanation, award **1 point**.

Problem 4/1/33:

Construction solutions

2 point: Student finds lattice points $A, B, C \in S$ such that $[ABC] = \frac{k}{2}$ for $1 \leq k \leq mn$ with sufficient supporting description. For example, it should be clear that the three points are in S. Award **1 point** of partial credit if the supporting description is insufficient.

1 point: Student shows that $[ABC] = \frac{k}{2}$ (e.g., standard area formula, Pick's Theorem, Shoelace Theorem).

2 points: Student shows how every desired area is attainable. Award **1 point** of partial credit for significant constructive progress towards this result.

Induction solutions

1 point: Student presents a valid base case.

1 point: Student states a valid inductive hypothesis.

3 points: Student completes the inductive step. For example, if we are inducting on m, this involves showing that all the areas from $\frac{(m-1)\cdot n+1}{2}$ to $\frac{mn}{2}$ inclusive are attainable. Award (at least) **1 point** of partial credit if the student shows a construction that obtains at least one area from $\frac{(m-1)\cdot n+1}{2}$ to $\frac{mn-1}{2}$ inclusive. Award **2 points** of partial credit if the inductive step is complete and correct except for a minor flaw.

Problem 5/1/33:

Note: Part (a) is worth **2 points** and part (b) is worth **3 points**.

Part (a): Award **2 points** for a complete and correct solution. An important intermediate step is recognizing that a key relationship is constant, so give **1 point** of partial credit for this result if the student attempts to make use of it in a meaningful way. In other words, simply writing $x_n x_{n-2} - x_{n-1}^2 = 5$ isn't enough to get points. The student needs to do something with this result, such as noticing that $x_n x_{n-2} - x_{n-1}^2 = x_{n+1} x_{n-1} - x_n^2$.

Part (b): Award **3 points** for a complete and correct solution. Award (at least) **1 point** of partial credit for any key intermediate result (such as our Lemma or $x_n = L_{2n}$) with proof or a valid citation and the remaining **2 points** for applying the result to solve the problem. Award partial credit as appropriate for significant constructive progress, keeping in mind how close or far the student is from a complete and correct solution. Other examples of useful intermediate results include $x_{-n} = x_n$ for all n and $x_n | x_{mn}$ for any odd integer m.

Note: Some students noticed that $x_n = L_{2n}$, where L_k is the k^{th} Lucas number. Many of these students cited a page on Lucas numbers from primes.utm.edu. Students who cited this page can get full credit as long as they explain how they apply the cited results to solve this problem. Citations of OEIS are also considered valid. If no proof or citation is provided, do not award credit for the corresponding claims.

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