1/1/18. When we perform a 'digit slide' on a number, we move its units digit to the front of the number. For example, the result of a 'digit slide' on 6471 is 1647. What is the smallest positive integer with 4 as its units digit such that the result of a 'digit slide' on the number equals 4 times the number?

Credit This problem was proposed by Naoki Sato.

Comments Let n be the number in the problem. Since the last digit of n is 4, the last digit of $4n$ is the same as the last digit of $4 \cdot 4 = 16$. But $4n$ is also the number obtained by performing a digit slide on n, so the last two digits of n are 64. One may repeat this process to find all the digits of n. *Solutions edited by Naoki Sato.*

Solution 1 by: Caroline Suen (11/CA)

Letting X be the positive integer in question, $4X$ is the result of the digit slide on X. The units digit of X is 4, and $4 \cdot 4 = 16$, so the units digit of $4X$ is 6, and the last two digits of X are 64.

We can continue the argument as follows:

$64 \cdot 4 = 256$, so the last two digits of $4X$ are 56, and the last three digits of X are 564,

$564 \cdot 4 = 2256$, so the last three digits of $4X$ are 256, and the last four digits of X are 2564,

$2564 \cdot 4 = 10256$, so the last four digits of $4X$ are 0256, and the last five digits of X are 02564,

$02564 \cdot 4 = 10256$, so the last five digits of $4X$ are 10256, and the last six digits of X are 102564, and finally

$102564 \cdot 4 = 410256$, which just happens to be the result of a digit slide on 102564.

Hence, 102564 is the smallest positive integer with 4 as its units digit such that the result of a digit slide on the number equals 4 times the number.

Solution 2 by: Howard Tong (11/GA)

Let x be the number formed by the digits other than the digit 4, and let x have k digits. Then the original number is $10x+4$, and the number obtained from the digit slide is $4 \cdot 10^k + x$. Therefore,

$$4 \cdot (10x + 4) = 4 \cdot 10^k + x,$$

which implies that

$$39x = 4 \cdot 10^k - 16.$$

The RHS is not divisible by 39 for $k = 1$, 2, 3, or 4, but when $k = 5$, $39x = 4 \cdot 10^5 - 16 = 399984 \Rightarrow x = 10256$. Therefore, the smallest possible number is 102564.

Solution 3 by: Shobhit Vishnoi (12/SC)

Let the number we are looking for be S. We have that $S = d_n d_{n-1} d_{n-2} \ldots d_2 4$, where each d_k represents a digit of the decimal expansion of S. Let us construct a rational repeating decimal number N, where

$$N = 0.d_n d_{n-1} d_{n-2} \ldots d_2 4 d_n d_{n-1} d_{n-2} \ldots .$$

By the conditions given in the problem, $4N$ must equal $0.4 d_n d_{n-1} d_{n-2} \ldots d_2 4 d_n d_{n-1} d_{n-2} \ldots$.

Thus, we have the following equations:

$$N = 0.d_n d_{n-1} d_{n-2} \ldots d_2 4 d_n d_{n-1} d_{n-2} \ldots, \quad (1)$$
$$4N = 0.4 d_n d_{n-1} d_{n-2} \ldots d_2 4 d_n d_{n-1} d_{n-2} \ldots . \quad (2)$$

Multiplying equation (2) by 10, we get

$$40N = 4.d_n d_{n-1} d_{n-2} \ldots d_2 4 d_n d_{n-1} d_{n-2} \ldots . \quad (3)$$

Subtracting equation (1) from equation (3) gives us $39N = 4$, so

$$N = \frac{4}{39} = 0.102564102564102564 \ldots = 0.\overline{102564}.$$

The repeating part of N is the desired number. Therefore, $S = 102564$. Checking, we see that $4 \cdot 102564 = 410256$, and indeed satisfies the conditions.

Additional Comments. This problem resembles problem 1 from the 1962 IMO:

Find the smallest natural number n which has the following properties:

(a) Its decimal representation has 6 as the last digit.

(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.