USA Mathematical Talent Search

Solutions to Problem 1/1/19

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1/1/19. Gene has $2n$ pieces of paper numbered 1 through $2n$. He removes n pieces of paper that are numbered consecutively. The sum of the numbers on the remaining pieces of paper is 1615. Find all possible values of n

Credit This problem was proposed by Richard Rusczyk.

Comments The first step in the problem is to use algebra to find suitable bounds on n. We can then use divisibility properties of integers to find the solutions. *Solutions edited by Naoki Sato.*

Solution 1 by: Carlos Dominguez (11/OH)

The minimum sum of the numbers on the remaining pieces is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, so $\frac{n(n+1)}{2} \leq 1615$. Clearing the denominator and expanding gives $n^2 + n \leq 3230$. Since $56^2 + 56 = 3192 < 3230$ and $57^2 + 57 = 3306 > 3230$, we must have $n \leq 56$.

The maximum sum of the numbers on the remaining pieces is $(n+1)+(n+2)+\cdots+2n = \frac{n(3n+1)}{2}$, so $\frac{n(3n+1)}{2} \geq 1615$, which implies $3n^2 + n \geq 3230$. Since $3 \cdot 32^2 + 32 = 3104 < 3230$ and $3 \cdot 33^2 + 33 = 3300 > 3230$, we must have $n \geq 33$.

Of the n numbers removed, let k be the first number. Then the sum of the n remaining numbers is

$$\begin{aligned} &(1 + 2 + \cdots + 2n) - [k + (k + 1) + \cdots + (k + n - 1)] \\ &= \frac{2n(2n + 1)}{2} - \frac{(2k + n - 1)n}{2} \\ &= 1615. \end{aligned}$$

Multiplying both sides by $2/n$ and expanding, we get

$$4n + 2 - (2k + n - 1) = 3n - 2k + 3 = \frac{3230}{n}.$$

Since $3n - 2k + 3$ is an integer, $3230/n$ is also an integer. In other words, n is a factor of 3230. The factors of 3230 are 1, 2, 5, 10, 17, 19, 34, 38, 85, 95, 170, 190, 323, 646, 1615, and 3230. The only factors between 33 and 56 (inclusive) are $n = 34$ and $n = 38$. The corresponding values of k are 5 and 16, respectively, which are both viable, so the possible values of n are 34 and 38.