
2/1/16. For the equation

$$(3x^2+y^2-4y-17)^3-(2x^2+2y^2-4y-6)^3=(x^2-y^2-11)^3,$$

determine its solutions (x,y) where both x and y are integers. Prove that your answer lists all the integer solutions.

Credit The original version of the problem was invented by Dr. George Berzsenyi, the creator of the USAMTS, and was modified into its current form by Dr. Erin Schram of NSA.

Comments Nearly all correct solutions followed one of the three strategies outlined in the published solutions below. Solutions 1 and 2 by Tony Liu and Zachary Abel exhibit two methods using substitution and algebraic manipulation. Solution 3 by Meir Lakhovsky employs Fermat's Last Theorem.

Solution 1 by: Tony Liu (10/IL)
To simplify the algebra, let us denote

$$\begin{aligned} a &= 3x^2+y^2-4y-17 \\ b &= 2x^2+2y^2-4y-6 \end{aligned}$$

so that

$$a-b = x^2-y^2-11$$

It follows that the original equation is equivalent to

$$\begin{aligned} a^3-b^3 &= (a-b)^3 \\ a^3-b^3 &= a^3-3a^2b+3ab^2-b^3 \\ 0 &= -3a^2b+3ab^2 \\ 0 &= 3ab(b-a) \end{aligned}$$

Now, we will have solutions if and only if $a=0$, $b=0$, or $a=b$.

Case 1: If $a=0$, then

$$\begin{aligned} 3x^2+y^2-4y-17 &= 0 \\ 3x^2+(y-2)^2 &= 21 \end{aligned}$$

Because the right hand side is divisible by 3, we conclude that $3|(y-2)^2$. Since squares are nonnegative, it follows that $(y-2)^2=0$ or 9. If $(y-2)^2=0$, then we have $3x^2=21$, which has no integral solutions. If $(y-2)^2=9$, then $y=-1$ or 5, and $3x^2=12$, so $x=\pm 2$. Our solutions (x,y) are thus

$$(-2,-1) \quad (-2,5) \quad (2,-1) \quad (2,5)$$

Case 2: If $b=0$, then

$$\begin{aligned} 2x^2+2y^2-4y-6 &= 0 \\ x^2+y^2-2y-3 &= 0 \\ x^2+(y-1)^2 &= 4 \end{aligned}$$

Since the only squares less than or equal to 4 are $0, 1$, and 4, we must have one term equal to 0 and the other equal to 4. If $x^2=0$, then $(y-1)^2=4$, and $y=-1$ or 3. If $x^2=4$, then $x=\pm 2$ and $(y-1)^2=0$, so $y=1$. Thus our solutions for this case are

$$(0,-1) \quad (0,3) \quad (-2,1) \quad (2,1)$$

Case 3: If $a=b$, then

$$\begin{aligned} 3x^2+y^2-4y-17 &= 2x^2+2y^2-4y-6 \\ x^2-y^2 &= 11 \\ (x+y)(x-y) &= 11 \end{aligned}$$

From this equation, we note that 11 is prime to conclude that $x+y=\pm 1$ or ± 11, and $x-y=\pm 11$ or ± 1. Solving these four systems of equations, we obtain the solutions

$$(-6,-5) \quad (-6,5) \quad (6,-5) \quad (6,5)$$

Our final solution list is thus

$$\begin{array}{cccc} (-2,-1) & (-2,5) & (2,-1) & (2,5) \\ (0,-1) & (0,3) & (-2,1) & (2,1) \\ (-6,-5) & (-6,5) & (6,-5) & (6,5) \end{array}$$

Since we have considered all of the cases, these are indeed all the solutions. ■

Solution 2 by: Zachary Abel (11/TX)
If we let $a = x^2 - y^2 - 11$ and $b = 2x^2 + 2y^2 - 4y - 6$, then we (conveniently) have $a + b = 3x^2 + y^2 - 4y - 17$. So the given equation becomes

$$0 = (a+b)^3 - a^3 - b^3 = 3ab(a+b)$$

Thus, either $a = 0$, $b = 0$, or $a + b = 0$. We do these cases separately.

Case 1: $\mathbf{a = 0}$.
We have $(x-y)(x+y) = 11$. But since 11 decomposes into the product of 2 integers in only four ways, we obtain the following systems of equations:

$$\begin{cases} x-y &= 1 \\ x+y &= 11 \end{cases} ; \quad \begin{cases} x-y &= 11 \\ x+y &= 1 \end{cases} ; \quad \begin{cases} x-y &= -1 \\ x+y &= -11 \end{cases} ; \quad \begin{cases} x-y &= -11 \\ x+y &= -1 \end{cases}$$

These systems give four solutions: $(x, y) = (6, 5)$, $(6, -5)$, $(-6, 5)$, and $(-6, -5)$.

Case 2: $\mathbf{b = 0}$.
The equation $2x^2 + 2y^2 - 4y - 6 = 0$ is equivalent to $x^2 + (y-1)^2 = 4$. There are only a few cases.

- If $|x| = 0$, then $(y-1)^2 = 4$, giving the solutions $(x, y) = (0, 3)$ and $(0, -1)$.
- If $|x| = 1$, then $(y-1)^2 = 3$, which is not solvable in integers.
- If $|x| = 2$, then $(y-1)^2 = 0$, giving the solutions $(x, y) = (2, 1)$ and $(-2, 1)$.
- If $|x| > 2$, then $0 \le (y-1)^2 = 4 - x^2 < 0$, which is impossible.

Case 3: $\mathbf{a + b = 0}$.
We have $3x^2 + y^2 - 4y - 17 = 0$, i.e. $3x^2 + (y-2)^2 = 21$. There are again 4 cases:

- If $|x| = 0$, then $(y-2)^2 = 21$, which is impossible in integers.
- If $|x| = 1$, then $(y-2)^2 = 18$, impossible in integers.
- If $|x| = 2$, then $(y-2)^2 = 9$, leading to $(x, y) = (2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$.
- If $|x| \ge 3$, then $(y-2)^2 = 21 - 3x^2 \le 21 - 3 \cdot 3^2 < 0$, which is impossible since squares are non-negative.

So there are 12 solutions: $(x, y) = (6, 5)$, $(6, -5)$, $(-6, 5)$, $(-6, -5)$, $(0, 3)$, $(0, -1)$, $(2, 1)$, $(-2, 1)$, $(2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$.

Solution 3 by: Meir Lakhovsky (9/WA)
Fermat's Last Theorem states that $a^n + b^n = c^n$ has integer solutions for $n \geq 3$ if and only if a, b, and/or $c = 0$. We let $(3x^2 + y^2 - 4y - 17) = c$; $(2x^2 + 2y^2 - 4y - 6) = b$; and $(x^2 - y^2 - 11) = a$, and note that our equation has the form $a^3 + b^3 = c^3$. Thus, either $[a = 0, b = c \neq 0]$ or $[b = 0, a = c \neq 0]$ or $[\ c = 0, a = -b \neq 0\ ]$ or $[\ a = b = c = 0\ ]$.

Case A: $a = 0, b = c \neq 0$
$x^2 - y^2 - 11 = 0 \Rightarrow (x - y)(x + y) = 11$. Since 11 is prime, its only factorizations are $11 * 1$ and $(-11) * (-1)$; therefore $(x - y)$ and $(x + y)$ equal either: $(1, 11)$ or $(11, 1)$ or $(-1, -11)$ or $(-11, -1)$ respectively. Solving each case individually, we get, $(6, 5)$, $(6, -5)$, $(-6, -5)$, and $(-6, 5)$ respectively. Furthermore, in all the cases $b = c \neq 0$. But, we notice, that whenever $a = 0$, we have $b \neq 0$, thus the case $a = b = c = 0$ is impossible.

Case B: $b = 0, a = c \neq 0$
$2x^2 + 2y^2 - 4y - 6 = 0 \Rightarrow x^2 + y^2 - 2y - 3 = 0 \Rightarrow x^2 + (y - 1)^2 = 4$. Since no two non-zero squares add up to 4, either x^2 or $(y-1)^2$ equal 0, while the other equals 4. Thus, the integral solutions in this case are $(2, 1)$, $(-2, 1)$, $(0, 3)$, and $(0, -1)$. All of these solutions give us $a = c \neq 0$, thus, all of them are valid.

Case C: $c = 0, a = -b \neq 0$
$3x^2 + y^2 - 4y - 17 \Rightarrow 3(x^2) + (y - 2)^2 = 21$. Through a little trial and error (plugging 0,1, 4, and for x^2), we see that the only possible value of x^2 which leaves y integral is 4. This yields the solutions $(2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$. In all of these cases $a = -b \neq 0$, thus, they are all valid.

There are no other integral solutions because we covered every case which follows from Fermat's Last Theorem. In conclusion, there are 12 solutions: $(6, 5)$, $(6, -5)$, $(-6, -5)$, $(-6, 5)$, $(2, 1)$, $(-2, 1)$, $(0, 3)$, $(0, -1)$, $(2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$.