2/1/17. George has six ropes. He chooses two of the twelve loose ends at random (possibly from the same rope), and ties them together, leaving ten loose ends. He again chooses two loose ends at random and joins them, and so on, until there are no loose ends. Find, with proof, the expected value of the number of loops George ends up with.

Credit This problem, or some variation, is often used in interviews for quantitative positions on Wall Street.

Comments The easiest approach, as the solution by James Sundstrom illustrates, is to develop a recursive formula for the number of expected loops formed when starting with n ropes. However, it is also possible to find the expected value by counting all possible loops. This was done in a clever way by Scott Kovach. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (11/NJ)

Let E_n denote the expected value of the number of loops from this process starting with n ropes. Then we have the following lemma:

Lemma. For all natural numbers n, $E_n = E_{n-1} + \frac{1}{2n-1}$.

Proof. If the process starts with n ropes, after one loose end is selected there are $2n-1$ loose ends remaining, giving a probability of $\frac{1}{2n-1}$ that the other end of the same rope will be selected as the second choice. If this occurs, there is one loop already formed and $n-1$ loose ropes left, so the expected value for the number of loops formed is $1 + E_{n-1}$. There is a probability of $\frac{2n-2}{2n-1}$ that an end of a different rope will be chosen, leaving $n-1$ ropes (1 longer one and $n-2$ short ones). Then the expected value of the number of loops is E_{n-1}. Combining the two possibilities gives:

$$\begin{aligned} E_n &= \frac{1}{2n-1} \times (1 + E_{n-1}) + \frac{2n-2}{2n-1} \times E_{n-1} \\ &= \frac{1}{2n-1} + \left(\frac{1}{2n-1} + \frac{2n-2}{2n-1} \right) \times E_{n-1} \\ &= E_{n-1} + \frac{1}{2n-1}. \end{aligned}$$

□

It is obvious that $E_1 = 1$, so we have:

$$E_2 = 1 + \frac{1}{3} = \frac{4}{3},$$
$$E_3 = \frac{4}{3} + \frac{1}{5} = \frac{23}{15},$$
$$E_4 = \frac{23}{15} + \frac{1}{7} = \frac{176}{105},$$
$$E_5 = \frac{176}{105} + \frac{1}{9} = \frac{563}{315},$$
$$E_6 = \frac{563}{315} + \frac{1}{11} = \frac{6508}{3465}.$$

Therefore, if George has six ropes which he ties together by randomly selecting two loose ends at a time to tie together, the expected value of the number of loops he will end with is $\frac{6508}{3465}$.

Solution 2 by: Scott Kovach (10/TN)

First consider the number of different ways to tie n ropes together. The first tie can be done in $\binom{2n}{2}$ ways, leaving $2n-2$ loose ends. The next can be done in $\binom{2n-2}{2}$ ways, the next in $\binom{2n-4}{2}$, and so on. The order that the ties are made doesn't matter, however, so we must divide the product of these binomial coefficients by $n!$. The number of ways therefore is

$$f(n) = \frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}}{n!} = \frac{\frac{(2n)!}{2(2n-2)!}\cdot\frac{(2n-2)!}{2(2n-4)!}\cdots\frac{2!}{2(0)!}}{n!} = \frac{(2n)!}{2^n n!}.$$

Now consider the number of ways to tie n ropes together to make a loop. The ropes can be sequentially tied to each other in any order, so there are $(n-1)!$ ways to order them. The first end of the first rope can be tied to either end of the second, the remaining end of the second to either end of the third, and so on. There are 2^{n-1} ways to do this, so the total number of ways to tie the n ropes together into a single loop is $g(n) = 2^{n-1}(n-1)!$.

Finally, we count the total number of loops among all the possible tyings. There are $\binom{6}{1}$ ways to choose one rope and $g(1)$ ways to tie it into one loop, and $f(5)$ ways to tie the remaining ropes together. Similarly, there are $\binom{6}{2}$ ways to choose two ropes, $g(2)$ ways to tie them into one loop, and $f(4)$ ways to tie the other four ropes together. Extending this process counts every possible loop of any size. The total number of loops is

$$\sum_{k=1}^{6}\binom{6}{k} g(k) f(6-k) = 19524.$$

There are $f(6) = 10395$ tyings, so the expected value is

$$\frac{19524}{10395} = \frac{6508}{3465}.$$